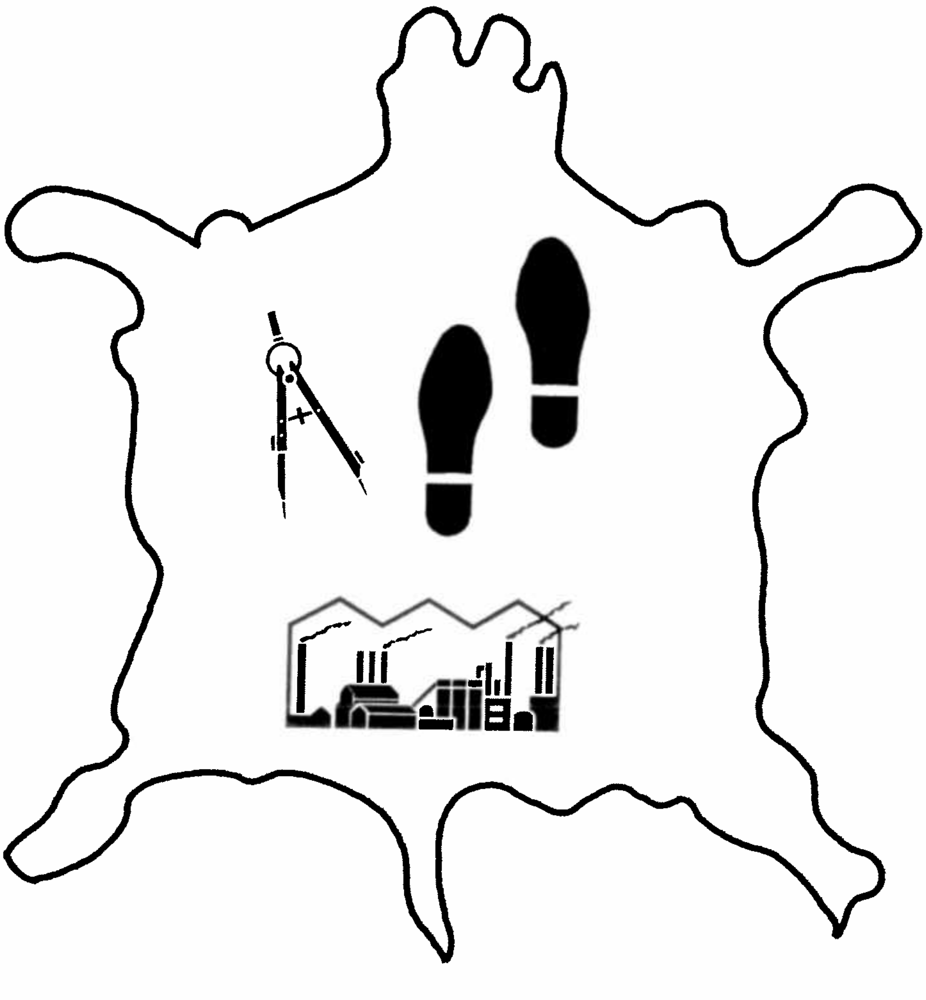
SHOE INDUSTRY CERTIFICATE COURSE

MATHEMATICS*



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This learning element was developed by the UNIDO Leather Unit's staff, its experts and the consultants of the Clothing and Footwear Institute (UK) for the project US/PHI/85/109 and is a part of a complete Footwear Industry Certificate course. The material is made available to other UNIDO projects and may be used by UNIDO experts as training aid and given, fully or partly, as hand-out for students and trainees.

The complete Certificate Course includes the following learning elements:

Certificate course

-Feet and last
-Basic design
-Pattern cutting
-Upper clicking
-Closing
-Making
-Textiles and synthetic materials
-Elastomers and plastomers
-Purchasing and storing
-Quality determination and control
-Elements of physics
-General management
-Production management
-Industrial Law
-Industrial accountancy
-Electricity and applied mechanics
-Economics
-SI metric system of measurement
-Marketing
-Mathematics
-Elements of chemistry

M A T H E M A T I C S

DESCRIPTION OF THE COURSE :

The course focuses on the application of mathematical principles, laws and theorems in the analysis of a given data. It involves representation of the data by mathematical notation or symbols, formulation of the necessary equations in the solution of the problems and computation of values of the unknown through long hand process, use of tables and calculating machines. Basic Mathematics, Algebra, and Trigonometry comprise the instructional areas of the course. Total

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FLIC/DIS Unit
June 1987

UNIT I BASIC MATHEMATICS

1.1 INTRODUCTION TO BASIC MATHEMATICS

RULES OF SIGNS

1-1.1 Commutative Property of Addition and Multiplication :

* The sum of any two real numbers is not affected by the order in which these numbers are added.

$$a + b = b + a$$

Example: $4 + 5 = 5 + 4$

* The product of any two real numbers is the same even if order of the factors is reversed.

$$a \times b = b \times a$$

Example: $2 \times 8 = 8 \times 2$

1-1.2 Associative Property of Addition and Multiplication

* The sum of any set of real numbers is not affected by the manner in which the numbers are grouped for addition. A pair of parentheses, ( ) can be used to group addends without changing their order and their sum

$$(a + b) + c = a + (b + c)$$

Example: $(5 + 6) + 8 = 5 + (6 + 8)$

$11 + 8 = 5 + 14$

$19 = 19$

* The product of any set of real numbers is not affected by the mannner in which the numbers are grouped for multiplication.

$$(a \times b) \times c = a \times (b \times c)$$

Example: $(8 \times 9) \times 10 = 8 \times (9 \times 10)$

$72 \times 10 = 8 \times 90$

$720 = 720$

1-1.3 Distributive Property of Multiplication Over Addition

* Multiplication is distributive over addition. This rule changes the product of two factors into the sum of two terms.

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$(b + c) \times a = (b \times a) + (c \times a)$$

The factor a in the products $a \times (b + c)$ and $(b + c) \times a$ can be distributed over the addends in the sum $b + c$.

Examples: (a) $6 \times (8 + 7) = (6 \times 8) + (6 \times 7)$

$6 \times 15 = 48 + 42$

$45 = 45$

Example: (b) $5 \times (3 + 6) = (5 \times 3) + (5 \times 6)$

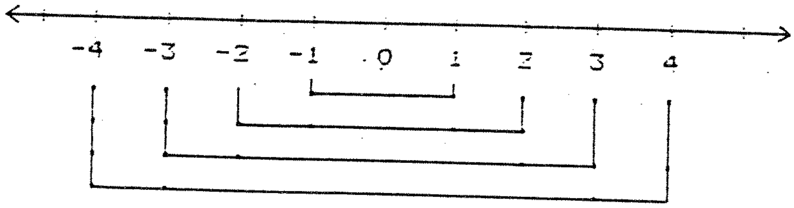
$5 \times 9 = 15 + 30$

$45 = 45$

1-1.2 FUNDAMENTAL OPERATIONS INVOLVING SIGNED NUMBERS

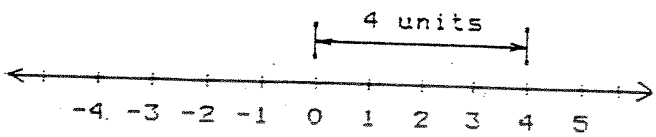
1-1.2.1 Positive and Negative Integers (Signed Numbers)

* Positive integers are numbers which may or may not be preceded by positive sign, +.
* If "a" is a real number, the symbol "-a" denotes the opposite or the additive inverse of "a" or the negative of "a". Every number may then be said to have an opposite.
* Numbers which are opposite of each other are best illustrated by a number line with zero as the initial reference point. A move to right of zero or an arrow pointing to the right represents a positive integer while a move to left of zero or an arrow pointing to the left represents a negative integer.

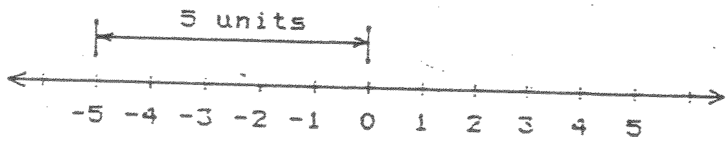


1-1.2.2 Addition of Signed Numbers

* Absolute value of a number is the distance of the number from the reference point, 0 .



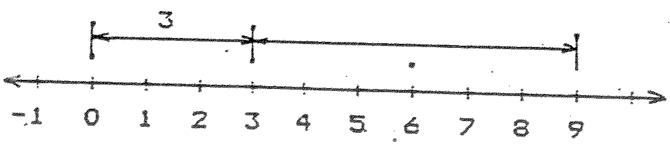
The absolute value of +4 = 4



The absolute value -5 = 5

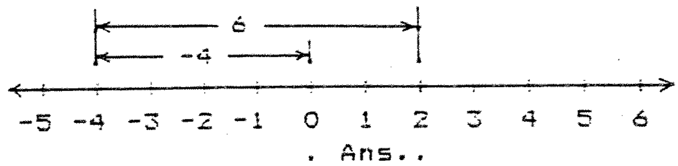
* Rule 1: To add numbers having like signs, add the absolute value of the terms and affix to the sum, the common sign.

3 + 6 = 9



* Rule 2: To add numbers having unlike signs, take the difference of their absolute values and prefix to the result the sign of the number with the bigger absolute value.

(-4) + 2 = 2



1-1.2.3 Subtraction of Signed Numbers

* Rule 3: To subtract a number from one another, mentally change the sign of the subtrahend and then proceed as in addition.

Thus if "a" and "b" are integers, then the difference, a-b is equal to the sum of "a" and the opposite of "b" . $a-b = a + (-b)$

Examples:

(a) $(-2) - 3 = (-2) + (-3) = -5$
(b) $7 - 4 = 7 + (-4) = 3$
(c) $2 - (-9) = 2 + (+9) = 11$
(d) $12 - (+5) = 12 + (-5) = 7$

1-1.2.4 Multiplication of Signed Numbers

* Rule 4: The product of two numbers with like signs is positive.

* Rule 5: The product of two numbers with unlike signs is negative.

Examples:

(a) $4 \times 3 = 12$
(b) $-4 \times (-3) = 12$
(c) $6 \times (-15) = -30$
(d) $(-8) \times (-3) = 24$
(e) $(+6) \times (+3) = 18$
(f) $(-9) \times 7 = -63$

1-1.2.5 Division of signed numbers

* Rule 6: Dividing two numbers with unlike signs produces a negative quotient.

Examples:

(a) $10 / 2 = 5$
(b) $-18 / 3 = -6$
(c) $-25 / (-5) = 5$

1-1.2.6 Order of Mathematical Operation

* When there are more than two operations involved and no signs of grouping exist in an expression, perform the multiplication and/ or division before any addition and/ or subtraction. Multiplication and division are of the same hierarchy. Thus, whichever of the two comes first (reading from left to right) should be performed first.

Examples:

(a) $(+6) - (+2) \times (-3) = 6 - (-6) = 6 + 6 = 12$
(b) $(-9) + 6 / 3 = -9 + 2 = -7$
(c) $5 \times (-7) + 8 / 4 = -35 + 2 = -33$
(d) $12 / 3 \times (-2) + (-4) = -12$

* If there are grouping symbols in an expression, simplify by eliminating the innermost pair. Perform the arithmetic operation within these sign of grouping. Repeat the process until all pairs of parentheses and brackets are gone.

Illustrative examples:

(a) $6 < 7 (8 - 5) + 3 (6 + 4) > = 6 < (3 \times 7) + (3 \times 10) >$
$= 6(21 + 30)$
$= 6 \times 51$
$= 306$

(b) $2 < 3 (6 - 4) + 4(6 + 4) > = 2 < (3 \times 2) + (4 \times 10) >$
$= 2(6 + 40)$
$= 2 \times 46$
$= 92$

(c) $< 16 (14 - 4) > - < 19 - (5 \times 2) >$
$= 16 \times 10 - (19 - 10)$
$= 160 - 9$
$= 151$

1-1.3 LOWEST COMMON MULTIPLE: HIGHEST COMMON FACTOR

1-1.3.1. Prime and Composite Numbers

* Prime numbers are numbers having 2 factors only, themselves and one.

Examples:

(a) $2 = 2 \times 1$
(b) $3 = 3 \times 1$
(c) $5 = 5 \times 1$
(d) $7 = 7 \times 1$
(e) $11 = 11 \times 1$
(f) $13 = 13 \times 1$
(f) $17 = 17 \times 1$
(g) $19 = 19 \times 1$

* Composite numbers are number having more than two factors.

Examples:

(a) 4 = 4 x 1
      = 2 x 2
(b) 6 = 2 x 2
      = 6 x 1
(c) 8 = 8 x 1
      = 4 x 2
      = 2 x 2 x 2
(d) 9 = 9 x 1
      = 3 x 3

* Prime factors of a number are prime numbers which when multiplied together yields the given number.

Examples:
(a) 15 = 5 x 3
(b) 18 = 3 x 3 x 2
(c) 36 = 3 x 3 x 2 x 2

1-1.3.2 Lowest Common Multiple

* Lowest common multiple or the least common multiple of a set of numbers is the smallest natural number for which all numbers is the set are factors. It is the number that is exactly divisible by all of them.

* Step 1: Write the numbers as products of prime factors
* Step 2. Choose the factors for the LCM, taking each which occurs in either or both.

Note: If the factors are the same for both numbers, it must be taken the number of times it appears most.

Examples:

(a)
15 = 3 x 5
18 = 3 x 3 x 2
LCM = 2 x 3 X 3 x 5 or 90

(b)
4 = 2 x 2
6 = 2 x 3
LCM = 2 X 2 X 3 OR 12

1-1.3.3 Highest Common Factor

* Highest common factor (HCF) or the greatest common factor (GCF) is the largest natural number which is a factor of all numbers in a given set. This is the highest value which can divide all numbers in the set exactly.

* To find the HCF of a set of numbers, the following steps are to be followed:

* Step 1: Write the numbers as product of their prime.

* Step 2: Choose the factor for GCF, by taking each factor which occurs in both.

Examples:

(a) 15 = 3 x 5
18 = 3 x 3 x 2
GCF is 3

(b) 4 = 2 x 2
6 = 2 x 3
GCF is 2

(c) 20 = 2 x 2 x 5
36 = 2 x 2 x 3 x 3
GCF is 4

1.2 FRACTIONS AND MIXED NUMBERS, RATIO, PROPORTION AND PERCENT

1-2.1.1 Fundamental Principles

Definition : Fraction is an indicated division. The quantity above the horizontal line is called numerator and that below the line is the denominator.

* Principle 1: The numerator and the denominator of a fraction can be multiplied by the same number or expression except zero, without changing the value of the fraction.

* Principle 2: The numerator and the denominator can be divided by the same number or expression, except zero, without changing the value of the fraction.

Examples:

(1) $\frac{2}{3} = \frac{2 \times 3}{3 \times 3} = \frac{6}{9}$

(2) $\frac{6}{9} = \frac{6 \div 3}{9 \div 3} = \frac{2}{3}$

1-2.1.2 Reduction of Fractions to Lowest Terms

* Rule: To reduce a fraction to its lowest term , factor the numerator and denominator into prime factors and cancel the factors common to both.

Note: Cancellation means divide both the numerator and the denominator by the common factors.

Examples:

(1) $27/108 = \frac{\cancel{3} \times \cancel{3} \times \cancel{3}}{2 \times 2 \times \cancel{3} \times \cancel{3} \times \cancel{3}} = \frac{1}{4}$

(2) $36/48 = \frac{\cancel{2} \times \cancel{2} \times \cancel{3} \times 3}{\cancel{2} \times \cancel{2} \times 2 \times 2 \times \cancel{3}} = \frac{3}{4}$

1-2.1.3 Signs of Fractions

* Rule 1: The sign before either term of a fraction can be changed if the sign before the fraction is changed.

* Rule 2: If the signs of both terms are changed, the sign before the fraction must not be changed.

Examples:

(1) $\frac{12}{13} = -\frac{-12}{13} = -\frac{12}{-13}$

(2) $\frac{3}{5} = +\frac{-3}{-5} = \frac{-3}{-5}$

1-2.1.4 Fundamental Operations

Addition and Subtraction of Fractions:

* Rule 1: To add two or more fractions having the same denominator, add their numerators and write the result over the common denominator.

Examples:

(1) $\frac{2}{7} + \frac{1}{7} + \frac{5}{7} = \frac{2+1+5}{7} = \frac{8}{7}$

(2) $\frac{4}{7} + \frac{5}{7} = \frac{9}{7}$

(3) $\frac{19}{15} + \frac{12}{15} = \frac{31}{15}$

* Rule 2: To subtract two fractions having the same denominator, subtract the numerator of the subtrahend from the numerator of the minuend and write the result over their common denominator.

Examples:

(1) $\frac{10}{11} - \frac{2}{11} = \frac{8}{11}$

(2) $\frac{6}{8} - \frac{1}{8} = \frac{5}{8}$

* Rule 3: To add or subtract fractions with different denominators,:

1. Reduce them to equivalent fractions with the least common denominator (LCD).
2. Combine (add or subtract as inndicated) the numerators of these equivalent fractions. The result written over the least common denominator is the answer.
3. Reduce the answer to lowest term.

Note: The LCD of a given set of dissimilar fraction is equivalent to the LCM of the denominators.

Examples:

(1) $\frac{3}{5} + \frac{1}{2} = \frac{6}{10} + \frac{5}{10}$

$\frac{11}{10}$

(2) $\frac{7}{4} - \frac{3}{6} = \frac{21}{12} - \frac{6}{12}$

$= \frac{15}{12}$

$= \frac{5}{4}$

(3) $\frac{4}{2} - \frac{8}{5} + \frac{7}{10} = \frac{20}{10} - \frac{16}{10} + \frac{7}{10}$

$= \frac{20 - 16 + 7}{10} = \frac{11}{10}$

Multiplication and Division of Fraction:

* Rule 1: The product of two fractions is the product of their numerators divided by the product of their denominators.

Examples:

(1) $\frac{1}{5} \times \frac{3}{2} = \frac{1\times3}{5\times2} = \frac{3}{10}$

(2) $\frac{12}{7} \times \frac{3}{2} = \frac{12\times3}{7\times2} = \frac{36}{14}$ or $\frac{18}{7}$

* Rule 2: To divide a fraction by another fraction, invert the divisor fraction and proceed as in multiplication of fractions.

Examples:

(1) $\frac{1}{7} \div \frac{3}{4} = \frac{1}{7} \times \frac{4}{3} = \frac{1\times4}{7\times3} = \frac{4}{21}$

(2) $\frac{5}{8} \div \frac{7}{6} = \frac{5}{8} \times \frac{6}{7} = \frac{5\times6}{8\times7} = \frac{30}{56} = \frac{15}{28}$

Addition and Subtraction of Mixed Numbers:

Definition:

Mixed numbers are numbers composed of a whole number and a fraction. Examples: 1 1/2, 5 3/4

* Rule 1: To add two or more mixed numbers, the sum is equal to the sum of the whole numbers added to the sum of the fractions.

Examples:

(1) 2 1/2 + 5 1/2 = 2 + 5 + 1/2 + 1/2 = 7+ 2/2
= 7+ 1 = 8

(2) 2 1/2 + 5 2/3 = 2+5+1/2+2/3 = 7+3/6+4/6
= 7+7/6 = 8 1/6

* Rule 2: To subtract two mixed numbers, the difference is equal to the difference between the whole numbers added to the difference between the fractions.

Examples:

(1) 8 3/8 - 5 2/8 = 8 3/8
-5 2/8
= 3 1/8

(2) 15 5/6 - 7 3/4 = 15 10/12
-7 9/12
= 8 1/12

(3) 10 1/2 - 7 7/8 = 10 4/8 = 9 12/8
-7 7/8 -7 7/8
= 2 5/8

Multiplication and Division of Mixed Numbers:

* Steps in multiplying two or more mixed numbers:

(1). Express the mixed numbers to improper fractions.
Examples:

(1) $5\ 3/4 = \frac{(4 \times 5) + 3}{4} = \frac{23}{4}$

(2) $6\ 1/2 = \frac{(2 \times 6) + 1}{2} = \frac{13}{2}$

(2). Find the product of the resulting inproper fractions by multipying the numerators and by multipying the denominators

Examples.

(1) $5\ 3/4 \times 6\ 1/2 = \frac{23}{4} \times \frac{13}{2} = \frac{23 \times 13}{4 \times 2} = \frac{299}{8}$ or $37\ 3/8$

* Steps in dividing mixed numbers:

(1). Express the mixed numbers to improper fractions.
(2) Find the quotient by inverting the divisor fraction and proceed as in multiplication of fraction.

Example:

(1) $5\ 3/4 \quad 6\ 1/2 = \frac{23}{4} \div \frac{13}{2} = \frac{23}{4} \times \frac{2}{13} = \frac{23 \times 2}{4 \times 13} = \frac{46}{52} = \frac{23}{26}$

UNIT II ELEMENTARY ALGEBRA

2-1 BASIC NOTATIONS AND RULES IN ADDITION AND SUBTRACTION

2-1.1 Definition of Terms:

a) Similar Terms - are terms with the same literal coefficients.

Examples: (1) $-4a$, $5a$

(2) xy^2, $6xy^2$

b) Monomial Expression - is an expression with one term.

Examples: (1) $3x$

(2) $-4ab$

c) Polynomial Expression - it is an expression with two or more terms separated by either (+) or (-) sign

Examples: 1.3 x 4b

(1) $b^2 - 6a + 4$

(2) $xy^2 + 2y - 4x + 7$

2-1.2 Addition and Subtraction of Algebraic Expressions:

Similar Terms:

* Rule 1 : To add or subtract similar terms, add or subtract their numerical coefficients and then retain the literal coefficients.

Examples: (1) $-19x + 3x - 2x = 18x$

(2) $2x^2y + 7x^2y - x^2y = 8x^2y$

Polynomials:

* Rule 2 : To add or subtract polynomials, arrange like terms in the same column and then apply rule 1.

Examples: (1) Add : $-3ab + 6cd + x^2y$; $14ab - 5x^2y$; $ab - 3cd$

Solution:

$$\begin{array}{lll} -3ab & + 6cd & + x^2y \\ 14ab & & -5x^2y \\ ab & - 3cd & \\ \hline 12ab & + 3cd & -4x^2y \end{array}$$

2-1.3 Symbols of Grouping:

To represent that two or more terms are to be considered as one quantity, symbols of grouping are used. These are the parentheses ( ), the bracket [ ], the braces { }, and the vinculuum $\overline{}$.

Removing the Symbols of Grouping:

* Rule 1 : Parentheses or other signs of grouping preceded by a plus sign can be removed without changing the sign of the terms.

Examples:

(1) $a + (b-c) = 1 + b - c$

(2) $(x+y) + (2x - 3y) = x + y + 2x - 3y$

* Rule 2 : To remove parentheses or other signs of grouping preceded by a minus sign, change the sign of every term within the sign of grouping.

Examples:

(1) $3a - (2b+c) = 3a - 2b - c$

(2) $10x - (-3x -4y) + 2y = 10x + 3x + 4y + 2y$

After using the two rules, combine similar terms and express the answer in simplest form.

Examples:

(1) $(x+y) + (2x-3y) = x + y + 2x - 3y$
$= 3x - 2y$

(2) $10x - (-3x - 4y) + 2y = 10x + 3x + 4y + 2y$
$= 13x + 6y$

* Rule 3 : To remove parentheses of other signs of grouping within grouping symbols in an expression, first perform the operation in the innermost pair and repeat the process until all pairs of parentheses or other signs of grouping are gone.

Examples:

$$\begin{aligned} & 3a - \left\{ 4 \left[3a + 6a\ (4a-b)\right]\right\} \\ = \; & 3a - \left\{ 4 \left[3a + 24a^2 - 6ab\right]\right\} \\ = \; & 3a - \left\{ 12a + 96a^2 - 24\ ab \right\} \\ = \; & 3a - 12a - 96a^2 + 24ab \\ = \; & -8a - 96a^2 + 24ab \end{aligned}$$

2.2. Multiplication and Algebraic Expression

2.2.1. Laws of Exponents in Multiplication

By definition $a^n = a.a.a.....$ to n factors

Where : n = exponent and is a positive integer

a = base

Example : Suppose $a^n = 2^3$

by definition of a^n what is t the value of 2^3 ?

$2^3 = 2.2.2 = 8$

Consider the product : $a^m a^n$

By definition this product is equal to m a's and n a's which is the product of m + n a's

So that : $a^m a^n = a^{m+n}$ where m and n are positive integers

By the definition of n^{th} power and by the commutative axiom $(ab)^n = a^n b^n = b^n a^n$ and $(a^n)^m = a^{mn}$

Example : 1. $(3x^4)^2 = 3^2 x^{4.2} = 9x^8$

2. $(xy)^2 = x^2 y^2$

3. $(x^2)\ (x^3) = x^5$

2.2.2. Law of Signs for Multiplication

From the following : $(-a)\ (-b) = ab$

$(a)\ (b) = ab$

$(-a)\ (b) = -ab$

$(a)\ (-b) = -ab$

it can be said that :

1. The product of two positive real numbers or of two negative real numbers is positive.
2. The product of a positive real number and a negative real number is negative.

$4x(7x) = 28x^2$

$-3a^2(-6b) = 18a^2b$

$6y^3(-9x^2) = -54y^3x^2$

$-5x^2(8x^3) = -40x^5$

2.2.3. Product of Algebraic Expressions

To obtain the product of two or more monomials, of a monomial and a polynomial and of two polynomials, the commutative, associative and distributive axioms together with the law of signs and the law of exponents are employed.

A. Products of Monomials and Polynomials.

The following examples are used to illustrate the method of obtaining the products of two or more monomials and a monomial and a polynomial :

$$3x^2y^2 . \ 4xy^2 . \ 6x^3y^4 = 3.4.6\ x^2.x.x^3.y^2.y^2.y^4$$
$$= 72x^{2+1+3}.y^{2+2+4}$$
$$= 72x^6y^8$$

$$-4ab^2c^3 . -2a^3b^4c . 6a^2bc^5 = -4.-2.6.a.a^3.a^2.b^2.b^4.b.c^3.c.c^5$$
$$= 48a^6b^7c^9$$

$$3ab(2a-4b+7a^2b) = 3ab(2a)-3ab(-4b)+3ab(7a^2b)$$
$$= 6a^2b + 12ab^2 + 21a^3b^2$$

$$(3x^2y-6xy^2-8y^3)(-5x^3y^2) = 3x^2y(-5x^3y^2) + (-6xy^2)(-5x^3y^2) + (-8y^3)(-5x^3y^2)$$
$$= -15x^5y^3 + 30x^4y^4 + 40x^3y^5$$

B. Product of two Polynomials.

To get the product of two polynomials the following steps are given in order :

1. First consider the second factor as a single number
2. Apply the right-hand distributive axiom
3. Complete the computation as indicated

Examples :

$$(-5x^2+2xy+3y^2)(3x^3-6x^2y+2xy-4y^3) = (-5x^2)(3x^3-6x^2y+2xy^2-4y^3)$$
$$+2xy(3x^3-6x^2y+2xy^2-4y^3)$$
$$+(3y^2)(3x^3-6x^2y+2xy^2-4y^3)$$
$$= -15x^5+30x^4y-10x^3y^2$$
$$+20x^2y^3+6x^4y-12x^3y^2+4x^2y^3$$
$$-8xy^4+9x^3y^2-18x^2y^3+6xy^4$$
$$-12y^5$$
$$= -15x^5+36x^4y-13x^3y^2+6x^2y^3$$
$$-2xy^4-12y^5$$

C. Products of two Binomials

Given are two binomials : $(ax + by)$ and $(cx + dy)$

Note that the terms of the two binominals are similar.

To get their product the right-hand distributive axiom and the left-hand distributive theorem are used, thus :

$$\begin{aligned}(ax + by)(cx + dy) &= ax(cx+dy) + by(cx+dy) \\ &= acx^2 + axdy + bcxy + bdy^2 \\ &= acx^2 + (ad+bc)xy + bdy^2\end{aligned}$$

From the above illustration the following steps are deduced :

1. Multiply the first terms in the binomials to obtain the first term in the product
2. Get the algebraic sum of the products obtained by multiplying the first term in each binomial by the second term in the other. This yieldsthe second term in the product.
3. Multiply the second terms in the binomials to get the third term in the product.

Ordinarily, these three steps can be done mentally.
Such that the product of $(ax+by)(cx+dy)$ can be written with no intermediate steps.

Example : $(2x-5y)(4x+3y) = 8x^2 - 14xy - 15y^2$

Get these products mentally

$2x.4x =$

$(2x.3y)+(-5y.4x) =$

$6xy - 20xy =$

$-5y.3y =$

2-3. SPECIAL PRODUCTS

2-3.1 Product of a Monomial and a Polynomial

This is equal to the sum of product of the monomial and each of the terms in the polynomial expression giving extra care to signs and exponents.

Examples:

1. $a^2cd\ (a - bc - d) \quad = \quad a^3cd - a^2bc^2d + a^2cd^2$
2. $3x^2y\ (5x^2 - 2xy + 3\ y^2) \quad = \quad 15\ x^4y - 6\ x^3y^2 + 9\ x^2y^3$
3. $-2\ m^2pq^3\ (5\ mp - 3\ pq^2 + 4\ m^2pq) \quad = \quad 10\ m^3p^2q^3 + 6\ m^2p^2q^5 - 8m^4p^2q^4$

2-3.2 Square of a Binomial

The product is equal to the sum of the square of the first term in the binomial, the product of the two terms taken twice, and the square of the second term.

Examples:

1. $(x + y)^2 \quad = \quad (x)^2 + 2(x)(y) + (y)^2 \quad = \quad x^2 + 2\ xy + y^2$
   2nd term
   1st term
2. $(x - y)^2 \quad = \quad (x)^2 + 2\ (x)(-y) + (-y)^2 \quad = \quad x^2 - 2\ xy + y^2$
   2nd term
   1st term
3. $(3a - 2b)^2 \quad = \quad (3a)^2 + 2\ (3a)(-2b) + (-2b)^2$
   $\quad = \quad 9a^2 - 12ab + 4b^2$

2-3.3 Product of a Sum and a Difference of Two Numbers

Let x be the first number and y, the other number.
Therefore,

Sum $= x + y$

Difference $= x - y$

Product $= (x + y)\ (x - y) \quad = \quad (x)^2 - (y)^2 \quad = \quad x^2 - y^2$

The product is equal to the difference of the square of the two numbers.

Examples:

1. $(3a - 2b)\ (3a + 2b) \quad = \quad (3a)^2 - (2b)^2 \quad = \quad 9a^2 - 4b^2$
2. $(7x^2 + 5y)\ (7x^2 - 5y) \quad = \quad (7x^2)^2 - (5y)^2 \quad = \quad 49\ x^4 - 25\ y^2$

2-3.4 Product of a Binomial and a Trinomial of three types:

$$* \quad \underbrace{(x + y)}_{\text{binomial}} \underbrace{(x^2 - xy + y^2)}_{\text{trinomial}} = x^3 + y^3$$

$$* \quad (x - y)(x^2 + xy + y^2) = x^3 - y^3$$

As indicated on the right side of the equality sign, the product is equal to the sum/difference of the cubes of the two terms in the binomial expression. Note: The first and the third terms in the trinomial factor are the squares of the first and second terms in the binomial factor, respectively. The negative of the product of the two terms in the binomial constitute the middle term of the trinomial.

Examples:

1. $(2x + 3y)(4x^2 - 6xy + 9y^2) = (2x)^3 + (3y)^3 = 8x^3 + 27y^3$
2. $(m^2 - 7p)(m^4 + 7m^2p + 49p^2) = (m^2)^3 - (7p)^3$
   $= m^6 - 343\ p^3$
3. $(5a - 2b)(25a^2 + 10\ ab + 4b^2) = (5a)^3 - (2b)^3$
   $= 125\ a^3 - 8b^3$

2-3.5 Square of Polynomial

The product of a square of a polynomial is equal to the sum of the squares of each term in the expression increased by the algebraic sum of twice the product of each term by every term that follows it.

Examples:

1. $(a - b + c + d - e)^2 = (a)^2 + (-b)^2 + (c)^2 + (d)^2 + (-e)^2$
   $+ 2(a)(-b) + 2(a)(c) + 2(a)(d)$
   $+ 2(a)(-e) + 2(-b)(c) + 2(-b)(d)$
   $+ 2(-b)(-e) + 2(c)(d) + 2(c)(-e)$
   $+ 2(d)(-e)$

   $(a-b + c + d - e)^2 = a^2 + b^2 + c^2 + d^2 + e^2 - 2ab + 2ac$
   $+ 2ad - 2ae - 2bc - 2bd + 2be + 2\ cd$
   $- 2ce - 2de$

   Terms that follow a: -b, +c, +d, and -e
   Terms that follow -b: +c, +d, and -e
   Terms that follow c: +d and -e
   Sole term that follows d: -e

2. $2p - 3q + 5t)^2 = (2p)^2 + (-3q)^2 + (5t)^2 + 2\,(2p)(-3q)$
$+ 2\,(2p)(5t) + 2\,(-3q)(5t)$
$= 4p^2 + 9q^2 + 25t^2 - 12pq + 20pt - 30qt$

Terms in the polynomial: 2p, -3q and 5t
Terms that follow <u>2p</u>: <u>-3q</u> and <u>5t</u>
Sole term that follows <u>-3q</u>: <u>5</u>

2-3.6 Expansion of a Binomial of the Type: $(x + y)^n$ or $(x - y)^n$ were <u>n</u> is a positive integer.

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3})y^3$$
$$+ \frac{n(n-1)(n-2)(n-3)}{4!}x^{n-4}y^4 + \ldots\ldots + nxy^{n-1} + y^n$$

$p! = p \text{ factorial} = (p)\,(p-1)\,(p-2)\,(p-3)\ \ldots\ 1$

2-3.6.1 Prop erties of the Binomial Expansion $(x \pm y)^n$:

1. The first term in the expansion $= x^n$
2. The second term $= nx^{n-1}y$
3. The esponent of x decreases by one and the exponent of y increases by one from term to term.
4. There are N+1 terms in the expansion.
5. The last term or (n+1)st term is y^n.
6. The second to the last term of the expansion $= nxy^{n-1}$
7. The sum of the exponents of x and y in any term is n.

Examples:

1. $(a-2b)^4 = (a)^4 + 4(a)^3\,(-2b) + \frac{(4)(3)}{(2)(1)}(a)^2(-2b)^2$
$+ \frac{(4)(3)(2)}{(3)(2)(1)}(a)(-2b)^3 + \frac{(4)(3)(2)(1)}{(4)(3)(2)(1)}(-2b)^4$

$(a-2b)^4 = a^4 + 4a^3(-2b) + 6a^2(4b^2) + 4a(-8b^3) + 16b^4$
$= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4$

2. $(2x + 5y)^3 = (2x)^3 + 3(2x)^2(+5y) + \frac{(3)(2)}{(3)(2)}(2x)(+5y)^2$
$+ \frac{(3)(2)(1)}{(3)(2)(1)}(+5y)^3$
$= 8x^3 + 3(4x^2)(+5y) + 3(2x)(25y^2) + 125y^3$
$= 8x^3 + 60x^2y + 150xy^2 + 125y^3$

Note that the signs of the terms in the expansion $(x+y)^n$ are all positive while that of $(x-y)^n$ are alternate positive and negative.

2-3.6.2 The rth term of the Binomial Expansion $(x \underline{+} y)^n$

$$\text{rth term} = \frac{n(n-1)(n-2) \quad \ldots. \quad (n-r+2)}{(r-1)!} \quad x^{n-r+1}y^{r-1}$$

Examples:

1. 6th term of $(a - 2b)^9$

$$r = 6$$
$$n = 9$$

$$\text{6th term} = \frac{(9)(8)(7)96)\overset{(9-6+2)}{(5)}}{(5)(4)(3)(2)(1)} \cdot (a)^{9-6+1}(-2b)^{6-1}$$

$$= 126\ a^4\ (-2b)^5 = 126\ a^4\ (-32b^5)$$
$$= -4032a^4b^5$$

2. 5th term of $(2x + 3y)^{10}$

$$r = 5$$
$$n = 10$$

$$\text{5th term} = \frac{(10)(9)(8)\overset{(10-5+2)}{(7)}}{(4)(3)(2)(1)} \cdot (2x)^{10-5+1}(3y)^{5-1}$$

$$= 210\ (2x)^6\ (3y)^4 = 210\ (64\ x^6)\ (81y^4)$$
$$= 1088640x^6y^4$$

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2-4 DIVISION OF ALGEBRAIC EXPRESSION

2-4.1 Law of Sign for Division:

The quotient of two positive or two negative numbers is positive. The quotient of a positive and a negative or a negative number and a positive number is negative.

Illustration:

a.) $\frac{12}{4} = 3$ b.) $\frac{24}{-6} = -4$ c.) $\frac{-18}{9} = -2$ d.) $\frac{-3}{-4} = +\frac{3}{4}$

2-4.2 Exponential Rule of Division

The quotient of a number raised ~~to~~ a certain power divided by the same number raised to another power is equal to that number raised to the difference between the powers of the dividend and the divisor.

Hence,

$$\frac{y^m}{y^n} = y^{m-n}$$

Illustration:

a.) $\frac{y^5}{y^2} = y^{5-2} = y^3$ b.) $\frac{15x^6}{-5x^4} = -3x^{6-4} = -3x^2$

2-4.3 Division of Multinomial by Monomial

To divide multinomial by a monomial, divide each term of the multinomial by a monomial and express the result as an algebraic sum.

Illustration:

Divide $3x^2y - 6xy^2 + 12x$ by $3x$

$$\frac{3x^2y - 6xy^2 + 12x}{3x} = \frac{3x^2y}{3x} - \frac{6xy^2}{3x} + \frac{12x}{3x} = xy - 2y^2 + 4$$

2-4.3.1 Division of Multinomial by Multinomial

Steps in dividing multinomial by a multinomial

1. Arrange the terms in the dividend and divisor in order of descending powers of a letter that appears in each.
2. Divide the first term in the dividend by the first term

in the divisor to get the first term in the quotient.

3. Multiply the divisor by the first term in the quotient and subtract the product from the dividend.
4. Treat the remainder obtained in step 3 as a new dividend and repeat step 2 and 3.
5. Continue this process until a remainder is obtained that is of lower degree than the divisor in the letter chosen in step 1 as the basis for the arrangement.

Note: A problem in division can be checked by using the relation

Dividend = Quotient x Divisor + Remainder

Illustration 1:

Divide $3x - 6x^2 + 18$ by $2x + 3$

$$
\begin{array}{lrl}
 & & -3x + 6 \quad \text{(Quotient)} \\
\text{(Divisor)} & 2x+3 \,\big)\!\! & \overline{-6x^2 + 3x + 18} \quad \text{(Dividend)} \\
 & & -6x^2 - 9x \\ \hline
 & & \qquad 12x + 18 \\
 & & \qquad 12x + 18 \\ \hline
 & & \qquad\quad 0
\end{array}
$$

Hence, we have

$$\frac{-6x^2 + 3x + 18}{2x + 3} = -3x + 6$$

Illustration 2:

Divide $6a^4 - 41a^2 + 3a + 6$ by $2a^3 - 4a - 3$

$$
\begin{array}{lrl}
 & & 3a^2 + 6a - 4 \quad \text{(Quotient)} \\
2a^2 - 4a - 3 & \big)\!\! & \overline{6a^4 \qquad - 41a^2 + 3a + 6} \quad \text{(Dividend)} \\
\text{(Divisor)} & & 6a^4 - 12a^3 - 9a^2 \\ \hline
 & & \quad 12a^3 - 32a^2 + 3a + 6 \\
 & & \quad 12a^3 - 24a^2 - 18a \\ \hline
 & & \qquad - 8a^2 + 21a + 6 \\
 & & \qquad - 8a^2 + 16a + 12 \\ \hline
 & & \qquad\qquad 5a - 6 \quad \text{(Remainder)}
\end{array}
$$

Hence, we have

$$\frac{6a^4 - 41a^2 + 3a + 6}{2a^2 - 4a - 3} = 3a^2 + 6a - 4 + \frac{3a - 6}{2a^2 - 4a - 3}$$

2-4.4 Synthetic Division

We can decrease the labor involved in the problem of finding the quotient and remainder when the polynomial in x is divided by the use of a process known as Synthetic Division.

In order to divide F(x) by x - r synthetically:

1. Arrange the coefficient of F(x) in order of descending powers of x, supplying zero as the coefficient of the missing power.
2. Replace the divisor x - r by +r
3. Bring down the coefficient of the largest power of x, multiply it by r, place the product beneath the coefficient of the second largest power of x, and add the product to that coefficient of the next largest power of x. Continue this procedure until there is a product added to the constant term
4. The last number in the third row is the remainder, f(r), and the other numbers, reading from left to right, are the coefficients of the quotient, which is of degree one less than F(x).

Illustration:

Determine the quotient and the remainder by dividing $2x^4 + x^3 - 16x^2 + 18$ by $x + 2$ synthetically.

Since $x - r = x + 2$, we have $r = -2$. Upon writing the coefficient of the dividend in a line, supplying zero as the coefficient of the missing term in x, and carrying out the steps of synthetic division, we have

$$\begin{array}{rrrrr|l} 2 & +1 & -16 & 0 & +18 & \underline{-2} \\ & -4 & +6 & +20 & -40 & \\ \hline 2 & -3 & -10 & +20 & -22 & \end{array}$$

Hence, the quotient is $2x^3 - 3x^2 - 10x + 20$ and the remainder is $f(-2) = -22$.

2.5 Algebraic Fractions - Multiplication and Division

2.5.1 The Fundamental Principle of Fractions

The value of the fraction is not changed when both its terms, i.e., the numerator and the denominator, are multiplied by or divided by the same number except zero.

If a fraction be denoted by $\frac{x}{y}$ where x is the numerator and y is the denominator, and if k is any number which is not equal to zero, then in accordance with the fundamental principle,

$$\frac{x}{y} = \frac{kx}{ky} = \frac{x \div k}{y \div k} , \qquad k \neq 0$$

2.5.2 Signs of a Fraction

There are three signs associated with a fraction namely: the sign of the numerator, the sign of the denominator and the sign of the fraction itself. Any two of these three signs can be change without changing the value of the fraction.

Example

(1) $$+\frac{-a}{-b} = -\frac{-a}{+b} = -\frac{+a}{-b} = +\frac{+a}{+b} = \frac{a}{b}$$

(2) $$\begin{aligned} +\frac{(x+a)(x-a)}{(a-x)} &= +\frac{(-x-a)(a-x)}{(a-x)} \\ &= -x-a \\ &= -(x+a) \quad \text{or,} \\ &= -\frac{(x+a)(a-x)}{(a-x)} \\ &= -\frac{(x+a)(x-a)}{(-a+x)} \\ &= -(x+a) \end{aligned}$$

2.5.3 Reduction of Fractions to Lowest Term

To reduce a fraction to its lowest term ;

1. Factor the terms of the fraction which are factorable
2. Apply the rule for changing the signs in a fraction if necessary.
3. Divide both numerator and denominator by the highest factor common to the terms. This is equivalent to the process of cancelling the same factors from the terms of the fraction.

Example 1

Reduce the following to lowest terms:

(a) $\frac{10}{15}$ (b) $\frac{-4x^2y}{8xy^2}$

Solutions:

(a) The highest factor common to the terms 10 and 15 is 5. Hence,

$$\frac{10}{15} = \frac{10 \div 5}{15 \div 5} = \frac{2}{3}$$

(b) The highest number which is a common factor to both $-4x^2y$ and $8xy^2$ is $4xy$; thus

$$\frac{-4x^2y}{8xy^2} = \frac{-4x^2y \div 4xy}{8xy^2 \div 4xy} = \frac{-x}{2y}$$

Example 2

Reduce the following to lowest term.

(a) $\frac{x^2 - 2x + 1}{x^2 + x - 2}$ (b) $\frac{x^3 - 8}{x^2 - 4}$

Solution: The highest number common as a factor to the terms of the given fraction is not immediately apparent in their present form. There is a need, therefore, to factor first the terms of the fraction. Hence,

a) $\dfrac{x^2 - 2x + 1}{x^2 + x - 2} = \dfrac{\cancel{(x-1)}(x-1)}{\cancel{(x-1)}(x+2)} = \dfrac{x-1}{x+2}$

b) $\dfrac{x^3 - 8}{x^2 - 4} = \dfrac{\cancel{(x-2)}(x^2 + 2x + 4)}{\cancel{(x-2)}(x+2)} = \dfrac{x^2 + 2x + 4}{x+2}$

2.5.4 Multiplication of Fractions

The product of two or more fractions is equal to the product of the numerators divided by the product of the denominators. Thus if $\frac{a}{b}$ and $\frac{c}{d}$ are any two fractions

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

Example 1

Multiply $\frac{2}{3}$ by $\frac{21}{4}$

Solution 1 $\quad \frac{2}{3} \cdot \frac{21}{4} = \frac{42}{12} = \frac{42 \div 6}{12 \div 6} = \frac{7}{2}$

Solution 2. $\quad \frac{2}{3} \cdot \frac{21}{4} = \frac{2}{3} \cdot \frac{3 \cdot 7}{2 \cdot 2} = \frac{\cancel{2}\,\cancel{3}\,7}{\cancel{3}\,\cancel{2}\,2} = \frac{7}{2}$

Example 2

Multiply $\dfrac{x^2 - 2x + 1}{x^2 - 1}$ by $\dfrac{x^3 + 1}{x^3 - 1}$

Solution $\quad \dfrac{x^2 - 2x + 1}{x^2 - 1} \cdot \dfrac{x^3 + 1}{x^3 - 1} = \dfrac{(x-1)(x-1)}{(x-1)(x+1)} \quad \dfrac{(x+1)(x^2 - x + 1)}{(x-1)(x^2 + x + 1)}$

$$= \frac{\cancel{(x-1)}\cancel{(x-1)}\cancel{(x+1)}(x^2 - x + 1)}{\cancel{(x-1)}\cancel{(x-1)}\cancel{(x+1)}(x^2 + x + 1)}$$

$$= \frac{x^2 - x + 1}{x^2 + x + 1}$$

Thus to multiply a fraction by one or more fractions;

1. Factor all the terms of the fractions which are factorable.
2. Apply the rule for changing signs in a fraction, if necessary.
3. Divide the terms of the fractions by the common factors.
4. The required product is equal to the products of all the remaining factors in the numerators divided by the product of all the remaining factors in the denominators.

2.5.5 Division of Fractions

The quotient of two numbers may be expressed as the product of the dividend by the reciprocal of the divisor.

$$a \div b = a \cdot \frac{1}{b} = \frac{a}{b}$$

where a and b are any two numbers provided $b \neq 0$. It follows from this therefore, that the quotient of two fractions is equal to the product of the fractional dividend and the reciprocal of the fractional divisor.

Example 1 $\quad \frac{4}{7} \div \frac{6}{5} = \frac{4}{7} \cdot \frac{5}{6} = \frac{2 \cdot 2}{7} \cdot \frac{5}{2 \cdot 3} = \frac{10}{21}$

Example 2

$$\frac{2x-4}{9} \div \frac{5x-10}{6} = \frac{2x-4}{9} \cdot \frac{6}{5x-10}$$

$$= \frac{2\cancel{(x-2)}}{3 \cdot 3} \cdot \frac{2 \cdot 3}{5\cancel{(x-2)}}$$

$$= \frac{4}{15}$$

Example 3

$$\frac{x^2+4x-12}{36-x^2} \div \frac{x^2-4x+4}{4-x^2} = \frac{x^2+4x-12}{36-x^2} \cdot \frac{4-x^2}{x^2-4x+4}$$

$$= \frac{(x + 6)(x - 2)}{(6 + x)(6 - x)} \cdot \frac{(2 + x)(2 - x)}{(x - 2)(x - 2)}$$

$$= \frac{-\cancel{(x + 6)}\cancel{(x - 2)}(2 + x)(2 - x)}{-\cancel{(6 + x)}(6 - x)\cancel{(x - 2)}(x - 2)}$$

$$= \frac{(x + 2)\cancel{(x - 2)}}{(x - 6)\cancel{(x - 2)}}$$

$$= \frac{x + 2}{x - 6}$$

2.6 SIMULTANEOUS LINEAR EQUATIONS

2.6 - 1 Simultaneous Linear Equations consists of two or more linear equations in the same unknowns and have a solution in common.

2.6 - 2 Graphical Solution of Two Linear Equations in Two Unknowns

Type of Equations	Distinguishing Features	Characteristic of Graphs	Solution
Dependent Equations	The 2 equations are equivalent. One of them could be determined by multiplying the Left and the Right members of the other equation by a certain constant.	They coincide.	Have infinite number of solutions. Each point that falls on both lines is a solution.
Inconsistent Equations	The Left member of one of the equations is a higher multiple of the Left member of the other.	They are parallel.	No solution at all. No point in common.
Consistent Equations	There is no definite pattern of relation between the Left and the Right members of the two equations.	They intersect at one and only one point.	Have one solution- the coordinates of the point where the two lines intersect.

Examples:

1. Dependent Equations

Eqn. 1. $2x-3y = 1$

Eqn. 2. $6x-9y = 3$

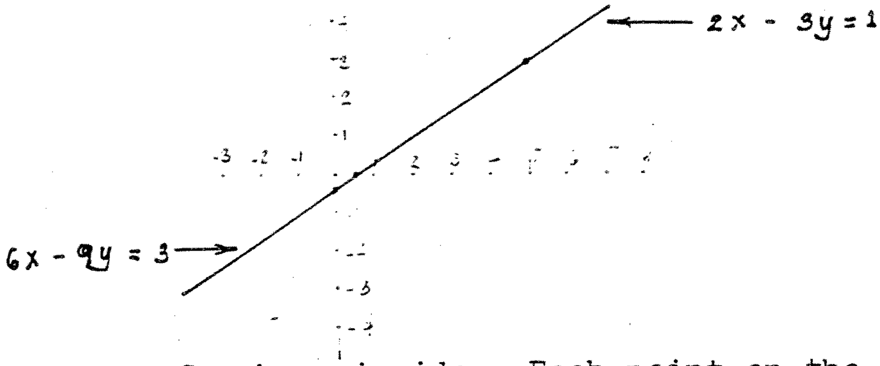
Distinguishing Features

3(Eqn. 1) $\Rightarrow$ Eqn. 2

$\Rightarrow 3[(2x-3y)=1]$

$\Rightarrow 6x-9y = 3$

Graphs and Solutions



Graphs coincide. Each point on the lines serves as a solution. And there are infinite number of solutions as the lines can be extended on both ends. Some of the solutions are (0,-1/3), (1/2,0), and (5,3).

2. Inconsistent Equations

Eqn. 1. $2x - y = 5$

Eqn. 2. $8x - 4y = 4$

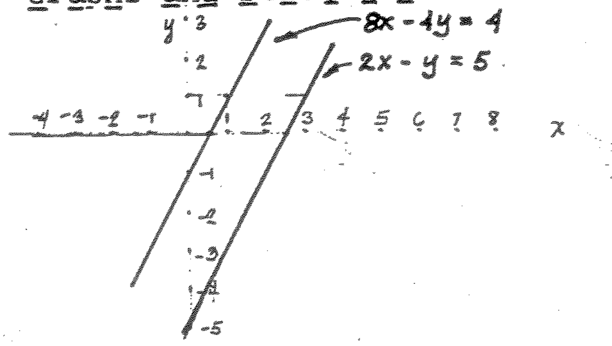
Distinguishing Features

$4\begin{bmatrix}\text{Left member}\\ \text{of Eqn. 1}\end{bmatrix} \Rightarrow \begin{bmatrix}\text{Left member}\\ \text{of Eqn. 2}\end{bmatrix}$

$\Rightarrow 4(2x - y)$

$\Rightarrow 8x - 4y$

Graphs and Solutions



Graphs are parallel. No Solution. There is no point in common.

3. Consistent Equations

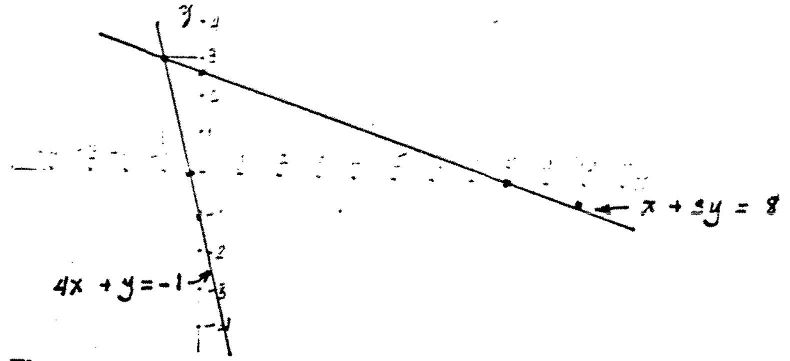
Eqn. 1 $4x + y = -1$

Eqn. 2. $x + 3y = 8$

Distinguishing Features

No definite pattern of relationship between the members of the two equations.

Graphs and Solution



There is one solution. It is located at the point of intersection of the lines. The coordinates of that point are x=-1, and y=3.

Refer to 2.9 for the procedure in constructing graphs.

2.6 -3.1 Analytical Solutions of Two Consistent Linear Equations in Two Unknowns

The three analytical methods forwarded here are:

Elimination of a variable by addition or subtraction,
Elimination of a varable
Elimination of a variable by substitution, and solution

by means of Determinants.

Hereafter, only consistent equations will be considered.

2.6.3 -3.1 Elimination of a Variable by Additon or Subtraction
STEPS:

1. Select the unknown that will be easier to eliminate. This is based on the coefficients of the unknowns.

2. Find the LCM of the two coefficients of the chosen unknown.

3. Multiply each equation by the quoffent of the LCM and the coefficient of the variable (step 1) in that equation.

4. Add/subtract the corresponding members of the equations obtained in step 3, depending on whether the terms containing the selected unknown have unlike or like signs.

5. Solve the resulting equation for the variable that remains.

6. Substitute the value obtained in step 5 in one of the given (original) equations and solve for the other unknown.

7. The solution , x = ___ and y= ___, is obtained by filling up the blanks with the values obtained in steps 5 and 6.

8. Check by substituting these values in the other original equation not used in step 6.

Example: Solve the following consistent equations by eliminating of a variable by additon/subtraction.

Equation 1. $3x - 2y = 16$

Equation 2. $2x + y = 6$

Solution:

Suppose x is selected to be eliminated. The LCM of the coefficients of x is 6. Divide 6 by 3 andthen by 2 to obtain 2 and 3, respectively.

$6x - 4y = 32$	Multiplying eqn. 1 by 2 (the quotient obtained by the LCM and the coefficent of x)→eqn. 3
$6x + 3y = 18$ (-) (-) (-)	Multiplying eqn. 2 by 3→eqn. 4.
$-7y = 14$	Subtracting eqn. 4 from eqn. 3.
$y = -2$	Dividing both members by -7.

By substituting y=-2 in eqn 1, we get

$3x - 2(-2) = 16$	
$3x + 4 = 16$	Performing the indicated multiplication.
$3x = 12$	Transposing and collecting terms
$x = 4$	Dividing both members of the eqn by 3.

Hence, the solution is x=4,y=-2.

Since eqn 1 was used in the process of obtaining the value of x, checking will be done in eqn 2. Substituting the obtained solution,

$2x + y = 6$

$2(4) + (-2) = 6$

$8 - 2 = 6$

The solution checks.

2.6 -3.2 Elimination of a Variable by Substitution

An alternative solution is by solving either of the equations for one of the unknowns in terms of the other and then substituing this value in the second equation.

STEPS:

1. For the sake of definitiveness, assume that the two equations are in terms of x and y unknowns. Solve one of the equations for x in terms of y.

2. Substitute the linear function of y obtained in step 1 for x in the second equation to obtain an equation that contains only the unknown y.

3. Solve the equation for y.

4. Substitute the value of y obtained in step 3 in the equation obtained in step 1 and then calculate the value of x.

5. The solution, x=___ and y=___, is obtained by filling up the blanks with the values obtained in steps 3 and 4.

6. Check by substituing the solution in the original equation not used in step 1.

Example: Solve for x and y by sustitution method.

Equation 1. $2x - 3y = -4$

Equation 2. $3x + y = 5$

Solution:

Solving for x in terms of y in equation 1,

$$2x - 3y = -4$$

$$2x = 3y - 4$$

$$x = \frac{3y - 4}{2}$$ Call this equation 3.

Substituting this value for x in equation 2 and then solving for the value of y,

$$3x + y = 5$$

$$3\left[\frac{3y - 4}{2}\right] + y = 5$$

$$\frac{9y - 12}{2} + y = 5$$

$$2\left[\frac{9y - 12}{2} + y\right] = 2[5]$$

$$9y - 12 + 2y = 10$$

$$11y = 22$$

$$y = 2$$

Substituting y=2 in equation 3;

$$x = \frac{3y - 4}{2}$$

$$= \frac{3(2) - 4}{2} \quad = \frac{6 - 4}{2} = \frac{2}{2} = 1$$

Check by substituting y=2 and x=1 in equation 2.

$$3x + y = 5$$

$$3(1) + 2 = 5$$

$$3 + 2 = 5$$

The solution checks.

2.6 -3.3 Solution of Two consistent Linear Equations by Means of Determinants

The preceding algebraic methods for solving systems of equations can be extended to three or more linear equations. But, the labor involved in such solutions becomes considerable as the number of equations in the system increases. At this point, the solution by determinants will be much of help.

Consider two consistent equations in x and y, Say,

$$ax + by = m \qquad \text{equation 1}$$
$$cx + dy = n \qquad \text{equation 2}$$

Define the square array of numbers (taken from the coefficients of x and y) to be equal to the product of the diagonal to the right minus the product of the diagonal to the left. Observe that the numbers are in the same relative positions they have in equations 1 and 2.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

D_{LEFT} D_{RIGHT}

This is known as the determinant of the coefficients. It is a determinant of the second order because it has two rows and two columns.

STEPS in Solving Any Pair of Consistent Linear Equations by Means of Determinants:

1. Transpose and arrange the terms in the equations so that the constant terms appear on the right and the terms involving the variables occur in the same order on the left. See forms of equations 1 and 2.
2. Express each solution (the value of each unknown) as the quotient of two determinants. In each case, the divisor is the determinant of the coefficients. The dividend in the value of each unknown is the determinant formed by replacing the coefficients of this unknown in the determinant of the coefficients by the constant terms, which in equations 1 and 2 are m and n.

NOTE: The corresponding constant terms replaced the coefficients of x.

$$x = \frac{\begin{vmatrix} m & b \\ n & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{md - bn}{ad - bc}$$

NOTE: The corresponding constant terms replaced the coefficients of y.

$$y = \frac{\begin{vmatrix} a & m \\ c & n \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \frac{an - mc}{ad - bc}$$

Example: Solve the set of equations

$2x - y = 4$

$3x - 4y = 1$

by determinants.

Solution: Notice that the equations are already in the form ax + by = constant. The determinant of the coefficients is:

$$D = \begin{vmatrix} 2 & -1 \\ 3 & -4 \end{vmatrix} = 2(-4) - (-1)(3) = (-8) - (-3) = -8+3 = -5$$

To obtain the dividend for x, replace the coefficients of x, 2 and 3, by the constant terms 4 and 1. Therefore,

$$x = \frac{\begin{vmatrix} 4 & -1 \\ 1 & -4 \end{vmatrix}}{D} = \frac{4(-4) - (-1)(1)}{-5} = \frac{(-16)-(-1)}{-5} = \frac{-15}{-5} = 3$$

Similarly,

$$y = \frac{\begin{vmatrix} 2 & 4 \\ 3 & 1 \end{vmatrix}}{D} = \frac{2(1)-(4)(3)}{-5} = \frac{2 - 12}{-5} = \frac{-10}{-5} = 2$$

The solution x=3, y=2 may be checked in either of the given equations by substitution.

An alternative way of solving for the value of y after obtaining the value of x by determinants is by substituting the value of x in either of the given equations, then calculating for y. Here, checking could be done using the other given equation.

2.6 - 3.4 Solution of a System of Three Linear Equations by Determinants (OPTIONAL)

Define the system of equations to be:

$$a_1x + b_1y + c_1z = d_1 \qquad \text{equation 1}$$

$$a_2x + b_2y + c_2z = d_2 \qquad \text{equation 2}$$

$$a_3x + b_3y + c_3z = d_3 \qquad \text{equation 3}$$

'Determine the determinant of the coefficients. Form the square array of numbers out of coefficients of the variables x, y, and z. This has a value e-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

qual to $a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3$. This is obtained by rewriting the first two columns and multipying the terms as indicated by the arrows. Sum up the product of the diagonals

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{array}{|ccc|cc} a_1 & b_1 & c_1 & a_1 & b_1 \\ a_2 & b_2 & c_2 & a_2 & b_2 \\ a_3 & b_3 & c_3 & a_3 & b_3 \\ - & - & - & + \; + & + \end{array}$$

to the right. Then subtract from this, the sum of all the product of the diagonals to the left. Note however, that the coefficients themselves have signs. Following the directions of the arrows, the value of the determinant of the coefficients is obtained as follows:

$$D = a_1b_2c_3 + b_1c_2a_3 + c_1a_2b_3 - c_1b_2a_3 - a_1c_2b_3 - b_1a_2c_3$$

This would give the same value as that of the above.

Hence,

NOTE: x coefficients replaced by the corresponding constant terms.

$$x = \frac{N_x}{D} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{D} \quad , \qquad D \neq 0$$

where:

N_x represents the determinant in the numerator of the value of x, and

D is the determinant of the coefficients.

Similarly,

NOTE: y coefficients replaced by the corresponding constant terms.

$$y = \frac{N_y}{D} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{D} \quad , \qquad D \neq 0$$

NOTE: z coefficients replaced by the corresponding constant terms.

$$z = \frac{N_z}{D} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{D} \quad , \qquad D \neq 0$$

Example: Solve

$$3x + 2y - z = 12 \qquad \text{equation 1}$$
$$x + y + z = 6 \qquad \text{equation 2}$$
$$x - 2y - z = -2 \qquad \text{equation 3}$$

simultaneously by use of determinants.

Solution:

(1) $D = \begin{vmatrix} 3 & 2 & -1 \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} \begin{matrix} 3 & 2 \\ 1 & 1 \\ 1 & -2 \end{matrix}$

$$D = (3)(1)(-1) + (2)(1)(1) + (-1)(1)(-2) - (-1)(1)(1)$$
$$-(3)(1)(-2) - (2)(1)(-1)$$
$$= -3 + 2 + 2 + 1 + 6 + 2$$
$$D = 10$$

(2) For the numerator of x, replace the coefficients of x in D by the constant terms.

$$N_x = \begin{vmatrix} 12 & 2 & -1 \\ 6 & 1 & 1 \\ -2 & -2 & -1 \end{vmatrix} \begin{matrix} 12 & 2 \\ 6 & 1 \\ -2 & -2 \end{matrix}$$

$$= (-12 - 4 + 12) - (2 - 24 - 12)$$
$$N_x = (-4) - (-34) = -4 + 34 = 30$$

(3) For the numerator of y, replace the coefficients of y in D by the constant terms.

$$N_y = \begin{vmatrix} 3 & 12 & -1 \\ 1 & 6 & 1 \\ 1 & -2 & -1 \end{vmatrix} \begin{matrix} 3 & 12 \\ 1 & 6 \\ 1 & -2 \end{matrix}$$

$$= (-18 + 12 + 2) - (-6 - 6 - 12)$$
$$= (-4) - (-24) = -4 + 24 = 20$$

(4) Therefore,

$$x = \frac{N_x}{D} = \frac{30}{10} = 3 \qquad \text{and} \qquad y = \frac{N_y}{D} = \frac{20}{10} = 2$$

(5) The value of z can also be determined by determinants but substituting the values of x and y in either of the original equations would be faster. Using equation 3 we get

$$3 - 2(2) - z = -2$$
$$3 - 4 - z = -2$$
$$z = +1$$

(6) Check by substitution in either of the other two equations.

UNIT III TRIGONOMETRY

M 3-1. Logarithm, Antilog, Use of Tables

3-1.1. Definition of Logarithm of a number

In $8 \ 2^3$, the relationship that exists between 8 and 3 is the exponent of the power to which 2 must be raised to produce 8.

The relationship between 3 and 8 can be indicated by the term "logarithm". Since 3 is the exponent to which 2 must be raised to obtain 8, then 3 is the logarithm of 8 to the base 2.

Therfore: "logarithm" of a positive number for a given base is the exponent that indicates the power to which the base must be raised in order to obtain the number.

Thus: $\log_b N = L$

where: $b^L = N$

Examples:

1. $\log_8 64 = 2$ since $8^2 = 64$

   $\log_4 64 = 3$ since $4^3 = 64$

   $\log_{81} 9 = \frac{1}{2}$ since $81^{\frac{1}{2}} = 9$

   $\log_a 1 = 0$ since $a^0 = 1$

2. Find the value of N if $\log_7 N = 2$

   $\log_7 N = 2$, indicates that $7^2 = N$
   therfore: $N \ 49$

3. If $\log_b 125 = 3$, find the value of b

   $b^3 = 125$

   $b = \sqrt[3]{125} = 5$ by extracting the cuberoot of both members to solve for b; by the law of exponents.

4. Find a if $\log_{27} 3 = 9$

   $27^a = 3$ or $3 = 27^a$
   $3 = (3^3)^a$ by the law of exponents
   Hence,it follows that $3a = 1$, then: $a = 1/3$

3-1.1.1. Characteristic and Mantissa

Reviewing: Scientific Notation

$$N = N'(10^c)$$

where: N = positive number not equal to 1
N' = is a number greater than or equal to 1, but less than 10 (1 N 10)
c = is an integer

Consider the following situations:

1. If $N \geq 10$, then $c \geq 1$
   Example, $231 = 2.31 \times 10^2$
2. If $N \leq 1$, then $c \leq 0$
   Example, $0.035 = 3.5 \times 10^{-2}$
3. If $1 < N < 10$, then, $N = N'$ and $c = 0$
   Example, $8.31 = 8.31 \times 10^0$, since $10^0 = 1$

Applying now the properties of logarithm, then:

$$\log N = \log N' + \log 10^c$$

$$= \log N' + c \text{ Log } 10$$

$$= \log N' + c, \text{ since } \log 10 = 1$$

Therfore: $\boxed{\log N = c + \log N'}$

Since $1 \leq N' < 10$, it follows that $10^0 \leq N < 10^1$,
Hence $0 \leq \log N' < 1$

From $\log N = c + \log N'$, it can be seen that the common logarithm of any positive number not equal to 1.

Then, the common logarithm of a positive number is expressed as an intege r plus a nonnegative decimal fraction, the integer is called the charaсteristic of the logarithm and the decimal fraction is called the mantissa.

In the expression for $\log N = c + \log N'$, the characteristic is the integer c and the mantissa is $\log N'$, where c is the exponent of 10 in the scientific notation for N.

The characteristic of the common logarithm of a positive number N is numerically equal to the number of digits between the reference position and the decimal point in N and is positive or negative according as the decimal point is to the right or to the left of the reference position.

3.1.2. Use of Table of Logarithm

3.1.2.1. Use of Tables to obtain the Mantissa

The mantissa of the logarithm of a number as previously discussed, is not affected by the position of the decimal point in the number.

Steps in Finding the Mantissas of three non-zero digit numbers:

1. Temporarily disregarding the decimal point, look for the first two digits in the table in the column headed by N.
2. In line with the first two digits under column N and across the page in the column headed by the third digit, find the entry.
3. Place a decimal point to the left of the entry, hence the mantissa of the logarithm of the number.

Examples:

1. Find the mantissa of the logarithm of 3.2 7.

From the table:

N	0	1	2	3	4	5	6	7	8	9
32								5145		

The mantissa of the logarithm of 3.27 is 0.5145 since, the decimal point in 3.27 is in the reference position, the characteristic of the logarithm is zero. Therefore: $\log 3.27 = 0.5145$.

2. Find the mantissa of the logarithm of 0.00634.

From the Table:

N	0	1	2	3	4	5	6	7	8	9
63					8021					

the mantissa of the logarithm (ml) of 0.00634 is 0.8021. Since the decimal point in 0.00634 is 3 places to the left of the reference position then the characteristic of the logarithm of 0.00634 is -3, which can be written as 7-10.

Therefore:

$$\log 0.00634 = 7.8021-10$$

Some Cases To be Considered when Finding Mantissa of the logarithm of a number:

1. If a number is composed of fewer than three digits, mentally annex one or two zeros at the right and proceed as before.

   Example: ml 52 = ml520, ml6 = ml600

2. If all the digits in a number after the third are zeros, disregard them in the process.

3. If a number is composed of four digits, the mantissa is obtained by a method known as linear interpolation.

   Example: 1. Find ml 412.8

   Since: 412.8 is between 412 and 413, ml 412.8 is between ml 412 and ml 413.

   From the table:

$$1\left[\begin{array}{l} \text{ml } 413 = 0.6160 \\ 0.8\left[\begin{array}{l}\text{ml } 412.8 = X \\ \text{ml } 412 = 0.6149\end{array}\right] x-0.6149 \end{array}\right] 0.0011$$

$$\frac{0.8}{1} = \frac{x-0.6149}{0.0011}$$

$$x-0.6149 = 0.8(0.0011)$$

$$x = 0.6149 + 0.0088 \} \ 0.0009$$

$$= 0.6158,$$ hence the mantissa of the log of 412.8

Example 2. Find the mantissa of logarithm of 0.006324.

Since the position of the decimal point has no effect on the mantissa, then:

$$ml\ 0.006324 = ml\ 6324$$

Note that 632.4 is between 632 and 633. Hence,

$$
1\left[\begin{array}{ll}
 & ml\ 633 = 0.8014 \\
0.4 & \left[\begin{array}{l} ml\ 632.4 = x \\ ml\ 632 = 0.8007 \end{array}\right] x-0.8007
\end{array}\right] 0.0007
$$

$$\frac{0.4}{1} = \frac{x-0.8007}{0.0007}$$

$$x = (0.4 \times 0.0007) + 0.8007$$

$$= 0.8010$$

Since the decimal point in 0.006324 is three places to the left of the reference position, then:

$$\log 0.006324 = 7.8010-10$$

From the foregoing examples the following a re the steps in interpolation:

1. Temporarily place the decimal point between the third and fourth digits of the given number.
2. Find the difference between the mantissa of the logarithms of the two 3-digit numbers between which the given number lies.
3. Muliply the difference between the two mantissas by the fourth digit of the given number considered as a decimal fraction.
4. Add the product obtained in step 3 to the smaller of the mantissas in step 2.

   *If a number contains more than four digits, round it off to four places and proceed as before.

3-1.2.2. Use of Tables to find N when log N is Given

The process is illustrated by the following examples:

1) Find N if log N = 1.6191

Procedure:

1. Find the mantissa 0.6191, in the body of the tables. Hence, in the tables find the mantissa starting with 61, then look through these until 6191 is located. 6191 is in line with 41 (in the column headed by N) and is in the column headed by 6. Thus, N is made up of the digits 416.
2. Place the decimal point. Since the characteristic of log N is 1, the decimal point is one place to the right of the reference position and hence is between 1 and 6. Therfore: N   41.6.

N	0	1	2	3	4	5	6	7	8	9
41							6191			

2) If ml N is not listed in the tables, then interpolation is done.

Steps:

1. Let T = number composed of the first four digits in N and shall determine T.
2. Place the decimal point by considering the characteristic and thus get N.

Example: Find N if log N = 5.4978

The mantissa 0.4978 is not listed in the table, but the two mantissas nearest to it are 0.4969 and 0.4983. These two mantissas are ml 3,140 and ml 3,150, respectively. (zero is added in each case in order to obtain four places for use in interpolation.)

$$\log N = 5.4978$$

$$14\left[\begin{array}{l} 0.4983 = \text{ml } 3{,}150 \\ 9\left[\begin{array}{l} 0.4978 = \text{ml } T \\ 0.4969 = \text{ml } 3140 \end{array}\right] T-3140 \end{array}\right] 10$$

By ratio and proportion:

$$\frac{9}{14} = \frac{T-3140}{10}$$

$$T-3140 = \frac{9}{14} \times 10$$

$$T = 3,140 + 6.4 = 3,146.4 = 3,146$$ round off, since the limit of accuracy is the fourth place.

Since the characteristic of log N is 5, then put the decimal point in N five places to the right of the reference position. Hence,

$$N = 314,600$$
$$= 3.146 \times 10^5$$

3.2. THE UNIT CIRCLE, TRIGONOMETRIC FUNCTIONS AND PLANE ANGLES

M 3-2.1 The Unit Circle

The circle whose radius is one unit long and whose center is the origin of a rectangular coordinate system is called the unit circle.

M 3-2.1.1 The Trigonometric Points

Start at the point A(1,0) on a unit circle and measure along the circumference an arc of length t units. If t is positive, measure the arc in the counterclockwise direction; it t is negative, measure the arc in the clockwise direction. This locates a unique point on the unit circle (Fig. A). We call the point thus located the trigonometric point P(t).

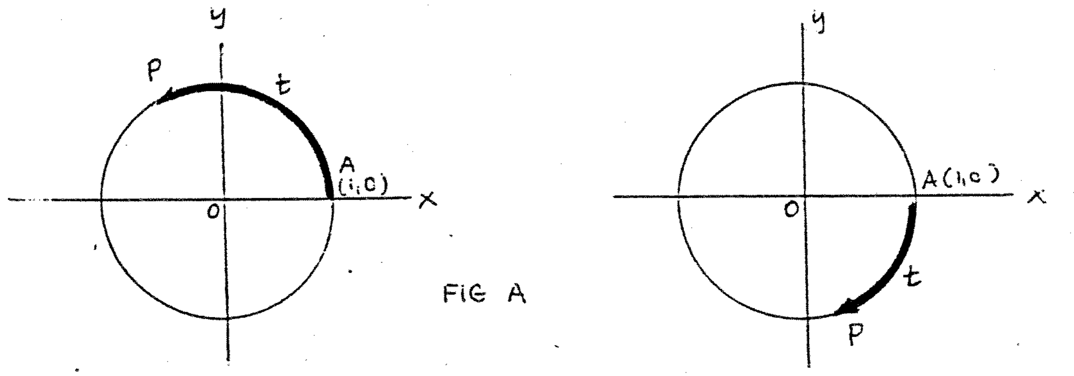


FIG A

When t = 0, P(0) is the trigonometric point A(1,0). Since the circumference of the unit circle is 2π units, one-half the circumference is π units, one-fourth the circumference is $\pi/2$ units, etc. Hence, when $t = \pi/2$, $P(\pi/2)$ is the point (0,1) Also, $P(\pi)$ is the point (-1,0); $P(3\pi/2)$ is the point (0,-1) (Fig. B). To locate trigonometric point, we let the coordinates be (x,y) and since P(t) is 1 unit from the origin, it follows that the equation of the circle is

$$x^2 + y^2 = 1$$

If one of the coordinates of the point P(t) is known, we can use the above equation to find the other coordinate.

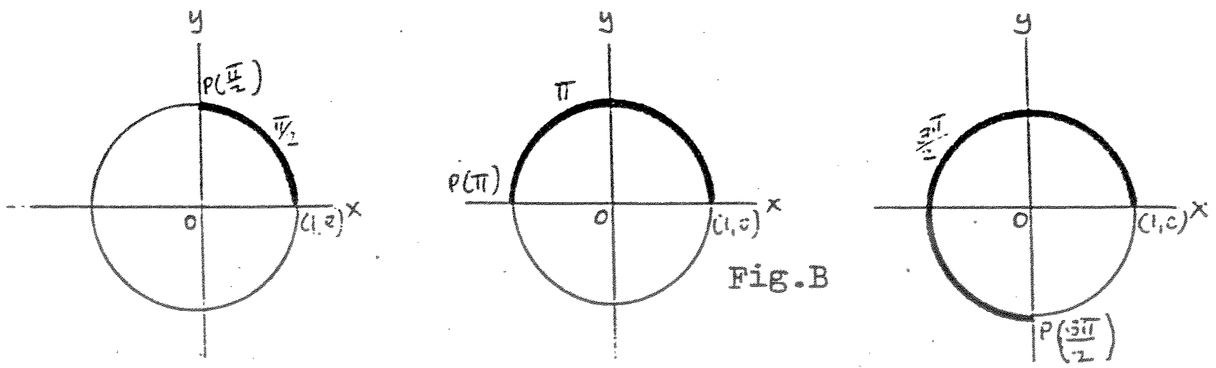


Fig.B

Example

Find the coordinate of the trigonometric point $P(\pi/4)$

From plane geometry, we see that the trigonometric point $P(\pi/4)$ is the midpoint of the arc on the unit circle from $(1,0)$ to $(0,1)$ and is equidistant from the x and y axes so that $x = y$ (see Fig. C). Substituting $y = x$ into the equation

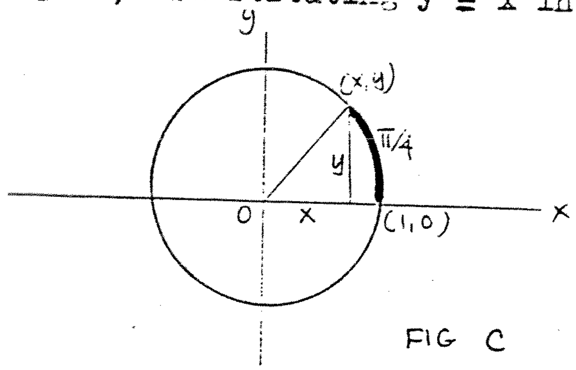


FIG C

$$x^2 + y^2 = 1 \text{ , we have}$$

$$x^2 + x^2 = 1$$

$$2x^2 = 1 \text{ , solving } x$$

$$x = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$y = x = \frac{\sqrt{2}}{2}$$

Therefore, the coordinates of the trigonometric point $P(\pi/4)$ are $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3-2.2 Trigonometric Functions

Let us consider the angle θ (Fig.D) which has been generated by rotating about the origin starting from coinsidence with OA. Take any point on its terminal side. With this point are associated three values; the abscissa x, the ordinate y and the radius vector r. Therefore we can define them with respect to the unit circle where $r = 1$, as follows

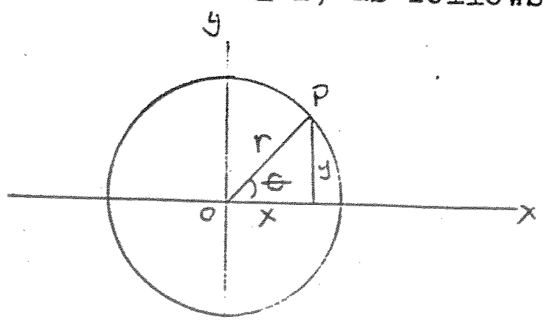


Fig.D

$$\text{Sine } \theta \ (\text{Sin } \theta) = \frac{\text{ordinate}}{\text{radius}} = \frac{y}{r} = y$$

$$\text{Cosine } \theta \ (\text{Cos } \theta) = \frac{\text{abscissa}}{\text{radius}} = \frac{x}{r} = x$$

$$\text{Tangent } \theta \ (\text{Tan } \theta) = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x}$$

$$\text{Cotangent } \theta \ (\text{Cot } \theta) = \frac{\text{abscissa}}{\text{ordiante}} = \frac{x}{y}$$

$$\text{Secant } \theta \ (\text{Sec } \theta) = \frac{\text{radius}}{\text{abscissa}} = \frac{r}{x} = \frac{1}{x}$$

$$\text{Cosecant } \theta \ (\text{Csc } \theta) = \frac{\text{radius}}{\text{ordinate}} = \frac{r}{y} = \frac{1}{y}$$

Since $x = \text{Cos } \theta$ and $y = \text{Sin } \theta$, it follows that the rectangular coordinate of any point P on the unit circle are ($\text{Cos } \theta$, $\text{Sin } \theta$). It will be noted that Tangent function and Cotangent function can also be defined respectively as

$$\text{Tangent } \theta - \frac{\text{Sin } \theta}{\text{Cos } \theta} \qquad \text{Cotangent } \theta - \frac{\text{Cos } \theta}{\text{Sin } \theta}$$

It is readily seen, from the generalized definition of the trigonometric functions that three of the functions are the reciprocal of the other three and we may write the Reciprocal Functions as

$$\text{Sin } \theta = \frac{1}{\text{Csc } \theta} \qquad \text{Cot } \theta = \frac{1}{\text{Tan } \theta}$$

$$\text{Cos } \theta = \frac{1}{\text{Sec } \theta} \qquad \text{Sec } \theta = \frac{1}{\text{Cos } \theta}$$

$$\text{Tan } \theta = \frac{1}{\text{Cot } \theta} \qquad \text{Csc } \theta = \frac{1}{\text{Sin } \theta}$$

Example Given $\sin \theta = \frac{3}{5}$, find the other functions of θ.

Solution: Since $\sin \theta = \frac{y}{r}$, we may take $r = 5$, from which it follows that $y = 3$. Draw a circle with its center at the origin and having a radius of 5 units (Fig.E). Take a point on the y-axis at a distance of 3 units above the x-axis. A line through this point parallel to the x-axis will cut the circle into two points, and concequently there will be two positions for the angle θ: θ_1 in quadrant I and θ_2 in quadrant II.

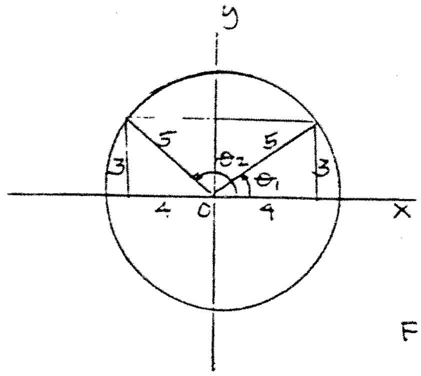


FIG E

Starting from the equation

$$x^2 + y^2 = r^2$$

$$x^2 = r^2 - y^2$$

$$x^2 = 5^2 - 3^2$$

$$x^2 = 25 - 9 = 16$$

$$x = \pm 4$$

Thus the corresponding to the angle in quadrant I we have an abscissa 4, and corresponding to the angle in quadrant II we have an abscissa -4. We can now read all the functions of both angles directly from the figure.

Quadrant I	Quadrant II
$\sin \theta_1 = \frac{3}{5}$	$\sin \theta_2 = \frac{3}{5}$
$\cos \theta_1 = \frac{4}{5}$	$\cos \theta_2 = -\frac{4}{5}$
$\tan \theta_1 = \frac{3}{4}$	$\tan \theta_2 = -\frac{3}{4}$
$\cot \theta_1 = \frac{4}{3}$	$\cot \theta_2 = -\frac{4}{3}$
$\sec \theta_1 = \frac{5}{4}$	$\sec \theta_2 = -\frac{5}{4}$
$\csc \theta_1 = \frac{5}{3}$	$\csc \theta_2 = \frac{5}{3}$

3-2.3 ARCLENGTH AND ANGLE

3-2.3.1 Angle

An angle is an amount of rotation of a half-line in a plane about its end point from an initial position to a terminal position.

Let O be the common end point of the two half-lines OA and OB (Fig F). The half-lines OA and OB are the sides of an angle, and the point O is the vertex of the angle.

FIG F

We now assign to this angle a number which corresponds to an "amount of rotation" in the plane of OA and OB about the vertex O which requires to make one of the half-lines coinsides with the other. The number is called the measure of the angle or simply the angle. The direction of the rotation is usaully indicated by a curved arrow from the initial side to the terminal side. If the rotation of the generating line is counterclockwise, the angle is positive (+); if the rotation of the generating line is clockwise, the angle is negative (-). If OA is rotated to coinside with OB, we call OA the initial side of the angle and OB the terminal side.

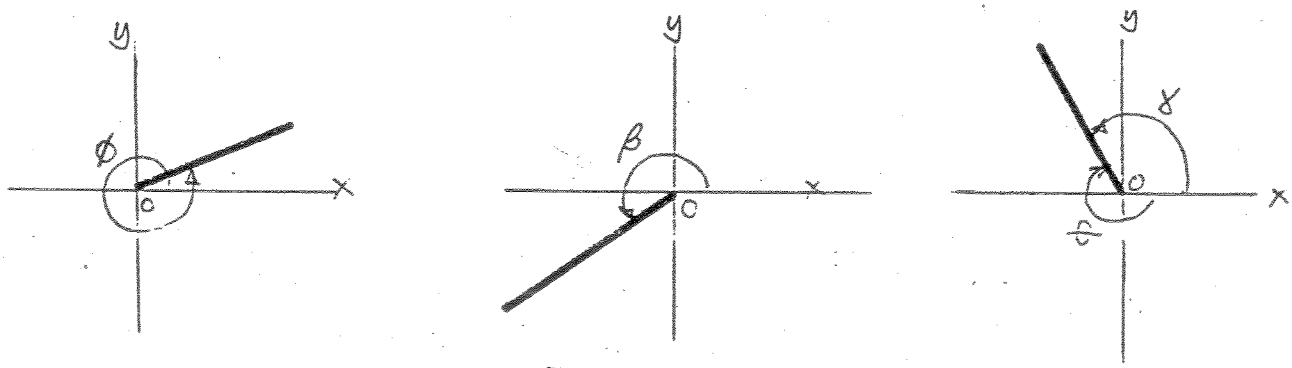


FIG G

When the vertex of an angle is the origin of a rectangular coordinate system and the initial side coinsides with the positive x axis, the angle is in "standard position". An angle in standard position is said to be in the quadrant in which the terminal side lies. Thus in Fig G. ϕ is in the first quadrant and β is in the third quadrant. An angle is called a "quadrantal angle" if it is in standard position and its terminal side coinsides with one of of the coordinate axes. Angles in standard position are called "Coterminal angles" if their terminal sides coinsides. In Fig G γ and θ are coterminal angles.

3-2.3.2 Degree And Radian Measure

Degree

A degree is defined to be an angle formed by a half-line rotated about its end point 1/360 of a complete rotation, or the measure of a central angle that substends an arc equal to 1/360 of the circumference of a circle. Thus,

$$360^{\circ} = 1 \text{ complete rotation}$$

$$180^{\circ} = 1/2 \text{ of a complete rotation}$$

$$90^{\circ} = 1/4 \text{ of a complete rotation}$$

and so on.

A minute, expressed by using ', is an angle formed by a rotation equal to 1/60 of a degree. Thus,

$$60' = 1 \text{ degree}$$

A second is an angle formed by a rotation equal to 1/60 of a minute and it is designated by ". Thus

$$60'' = 1 \text{ minute}$$

Radian

A radian is the angle formed by a half-line that has been rotated its end point $1/2\pi$ of a complete rotation, or the measure of a central angle of a circle which substends on the circumference an arc length equal to the length of its radius. Thus,

$$1 \text{ radian} = 1/2\pi \text{ complete rotations}$$

$$2 \text{ radian} = 2(1/2\pi) \text{ complete rotations}$$

$$3 \text{ radian} = 3(1/2\pi) \text{ complete rotations}$$

$$2\pi \text{ radian} = 2\pi(1/2\pi) \text{ complete rotations or 1 complete rotation.}$$

An angle of one radian is a central angle subtended by an arc
equal in length to the radius of the circle. Let angle AOB (Fig
H) be the central angle of the circle whose radius is r units
of length. Furthermore, let an angle AOB be an angle of 1 radian
and let the length of the intercepted arc be t units. Since the
the ratio of the measure of two central angles of a circle
equals the ratio of the lengths of their respective intercepted
arc, we have

Therefore $\quad 2.6$ rad $\doteq 148°58'1.2''$.

Example 3

Express 31^{o} in radians

Since $1^{o} = \pi/180$ rad

$$31^{o} = 31\ (\pi/180)$$
$$= \frac{31\pi}{180} \text{ rad}$$

3-2.3.3 Arc Length

Consider a circle whose radius OA has length r (Fig I) Let arc AB be an arc of length r. Then angle AOB is an angle of 1 rad. Let arc AC be an arc of length s and let θ equal the number of radians in angle of the measures of their respective intercepted arcs.

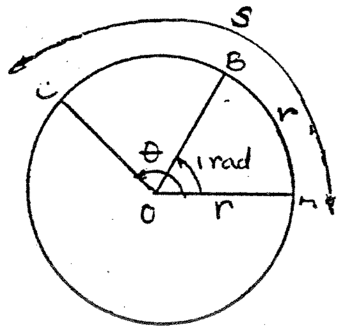


FIG I.

$$\frac{\text{measure of } \measuredangle AOB}{\text{measure of } \measuredangle AOC} = \frac{\text{length of arc AB}}{\text{length of arc AC}}$$

$$\frac{1}{\theta} = \frac{r}{s}$$

$$s = r\theta$$

Thus, the length of any circular arc can be found by multiplying the number of units in the length of the radius by the number of radians in the angle subtended by the arc.

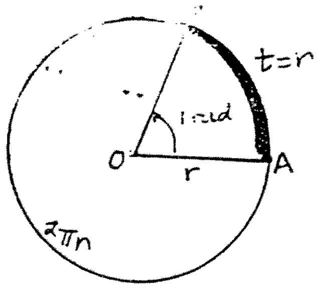
Example

The radius of a circle is 15 in. Find the length of an arc of this circle which subtends a central angle of 60^{o}.

The central angle $\theta = 60^{o} = \pi/3$ rad

hence, $\quad s = r\theta$

$$= 15\ (\pi/3) = 5\pi = 15.7$$



$$\frac{1/2 \text{ complete rotation}}{1 \text{ complete rotation}} = \frac{t}{2\pi r}$$

$$t = r$$

Since 360° = one complete rotation and also 2π rad = one complete rotation, we have the ff. relation

$$2\pi \text{ rad} = 360^\circ$$

$$\pi \text{ rad} = 180^\circ$$

$$\pi/2 \text{ rad} = 90^\circ$$

$$\pi/3 \text{ rad} = 60^\circ$$

and also

$$1 \text{ rad} = \frac{180}{\pi} = 57.295^\circ$$

$$1^\circ = \frac{\pi}{180} = 0.01745 \text{ rad}$$

Example 1

Express $7\pi/4$ rad in degrees

Since π rad = 180°, $7\pi/4$ rad $= (7/4)(180^\circ)$
$= 315^\circ$

Example 2

Express 2.6 rad in degrees, minutes and seconds

Since 1 rad = 57.295°

$$\begin{aligned} 2.6 \text{ rad} &= 2.6\ (57.295^\circ) \\ &= 148.967^\circ \\ &= 148^\circ + 0.967^\circ \end{aligned}$$

$$\begin{aligned} 0.967^\circ &= 0.967\ (60') \\ &= 58.02' \\ &= 58' + 0.02' = 58' + 0.02\ (60'') \\ &= 58' + 1.2'' \end{aligned}$$

3-2.3.4 Specific Angles based on Measures

The specific angles based on measures are special angles 30°, 45°, and 60°. The exact values of these angles are easily computed by using elementary geometry.

30 Degree Angle

An angle of 30° or $\pi/6$ radians (Fig J) is in the standard position. From point P where the terminal side intersects the unit circle, PQ is drawn perpendicular to the x-axis, forming right triangle OPQ. We know from plane geometry that in the $30^\circ-60^\circ-90^\circ$ triangle, the hypotenuse is twice the shorter side.

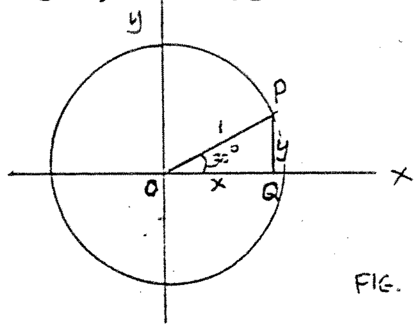


FIG. J

Since $r = 1$, then in right triangle OPQ, $y = 1/2$, substituting these values in the equation of the unit circle

$$x^2 + y^2 = 1$$

$$x^2 + (\tfrac{1}{2})^2 = 1$$

$$x^2 = 3/4$$

$$x = \sqrt{3}/2$$

We take the positive value of x because P is in the first quadrant. Hence, the coordinates of P are $(\sqrt{3}/2, 1/2)$ and the circular functions of 30° are

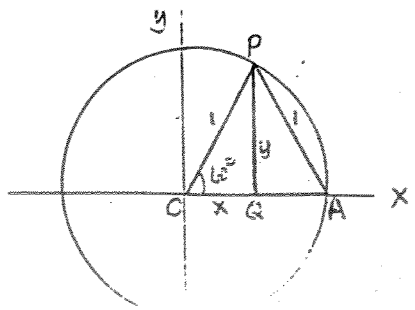
$\sin 30^\circ = 1/2$ $\quad\tan 30^\circ = 1/\sqrt{3}$ $\quad\sec 30^\circ = 2/\sqrt{3}$

$\cos 30^\circ = \sqrt{3}/2$ $\quad\cot 30^\circ = \sqrt{3}$ $\quad\csc 30^\circ = 2$

60 Degree Angle

The terminal side of an angle 60° or $\pi/3$ radians, (Fig K), in the standard position, intersects the unit circle at P. The chord AP is drawn, forming equilateral triangle OPA whose sides are each one unit long. Then the perpendicular bisector PQ of OA is drawn.

FIG. K



Hence, $x = 1/2$. By the equation $x^2 + y^2 = 1$ of the unit circle, we get $y = \sqrt{3}/2$. Therefore, the coordinates of P are $(1/2, \sqrt{3}/2)$, and the circular functions of 60° are

$$\sin 60^\circ = \sqrt{3}/2 \qquad \tan 60^\circ = \sqrt{3} \qquad \sec 60^\circ = 2$$
$$\cos 60^\circ = 1/2 \qquad \cot 60^\circ = 1/\sqrt{3} \qquad \csc 60^\circ = 2/\sqrt{3}$$

45 Degree Angle

In Fig L the terminal side of an angle 45° or $\pi/4$ radians, in the standard position, intersects the unit circle at P. PQ is drawn perpendicular to the x-axis, forming isosceles right triangle OPQ. Thus, $x = y$.

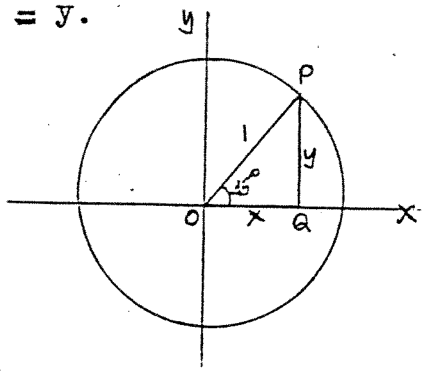


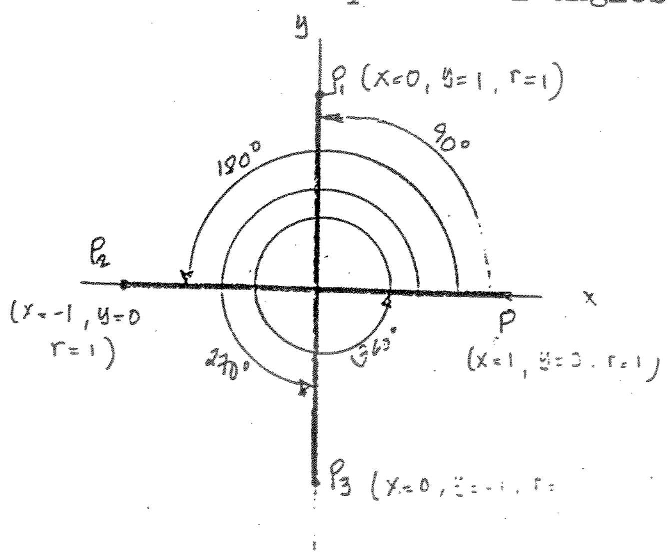
FIG. L

Since $x^2 + y^2 = 1$, we have $x = 1/\sqrt{2}$ and $y = 1/\sqrt{2}$. Hence, the coordinates of P are $(1/\sqrt{2}, 1/\sqrt{2})$ and the functions of 45° are

$$\sin 45^\circ = 1/\sqrt{2} \qquad \tan 45^\circ = 1 \qquad \sec 45^\circ = \sqrt{2}$$
$$\cos 45^\circ = 1/\sqrt{2} \qquad \cot 45^\circ = 1 \qquad \csc 45^\circ = \sqrt{2}$$

3-2.3.5 Quadrantal Location of an Angle

A quadrantal angle is an angle in the standard position whose terminal side falls on one of the axes. Angles 0°, 90°, 180°, 270° and 360° are quadrantal angles. Fig M



From Fig M, in which each of the points P, P_1, P_2 and P_3 is at a numerical distance of 1 unit from the origin, we can read off the functions of quadrantal angles namely $0^o, 90^o, 180^o$, 270^o and 360^o. The value of these functions are tabulated below.

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0^o	0	1	0	∞	1	∞
90^o	1	0	∞	0	∞	1
180^o	0	-1	0	∞	-1	∞
270^o	-1	0	∞	0	∞	-1
360^o	0	1	0	∞	1	∞

3.3. COMMON PLANE FIGURES

A closed broken line is called a polygon. For a figure to be a polygon, it must not only be bonded by line segments, but must also be closed and must be on one plane. A plane is a surface that wholly contains a straight line joining any two points in it.

The line segments bounding the polygon are called sides and the points where every two sides meet are called the vertices. A polygon may have three sides, four sides, five sides, six sides or any number of sides.

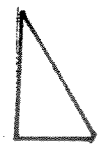
3.3.1. Features of the Plane Figures

3.3.1.1. Triangle A very important type of a polygon is a Triangle. This is a polygon of three sides and, as the name implies, three angles (triangle meaning "three angles".)

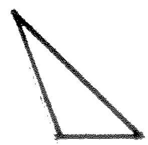
A very important property of all triangles is that the sum of the three angles is 180°. Therefore, it is clear that the triangle cannot have more than one right angle or one obtuse angle. At least two of the three angles are acute.

If the third angle is also acute, then the triangle is called an acute triangle ; if the third angle is a right angle, the triangle is called right triangle ; if the third angle is an obtuse angle, it is called obtuse triangle.

acute triangle



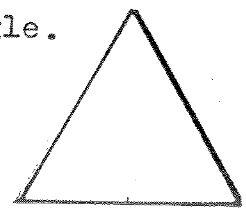
right triangle



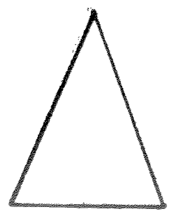
obtuse triangle

If the three angles of a triangle are all equal, all the three sides are the same in length. Such a triangle is called an equilateral triangle. (equi meaning equal, lateral = sides)

If two angles of a triangle are equal, the sides opposite these angles are equal. Such triangle is called isosceles triangle.



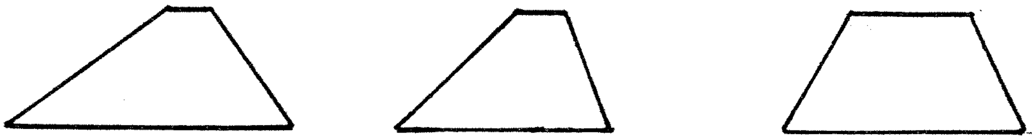
equilateral triangle



isosceles triangle

If we take one side of a triangle as the base, then the altitude of the triangle is the perpendicular drawn to that base from the opposite vertex.
Thus, in the adjoining triangle ABC, if AC is the base, then the perpendicular drawn to AC from the opposite vertex B is the altitude BD.

3.3.1.2. <u>Trapezoid</u>. A trapezoid is a quadrilateral (+) which has one and only one pair of parallel sides. If the non-parallel sides are equal, the trapezoid is called <u>isosceles</u>. In a trapezoid, the two parallel sides are called the bases, and the perpendicular from any point of one base to the other base is called the altitude of the trapezoid.

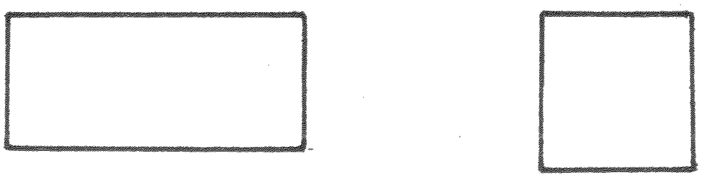


trapezoid   isosceles trapezoid

In the trapezoid at the left, AD and BC are the bases and BE the altitude.

3.3.1.3. <u>Rectangle</u>. A rectangle is a parallelogram (++) whose sides intersect in the right angles. The longer side of the rectangle is called the length while the shorter side is called the width.

3.3.1.4. <u>Square</u>. A square is a rectangle with all sides equal in length. A square is a special type of a rectangle. Since all sides are equal, then : length = width = sides.

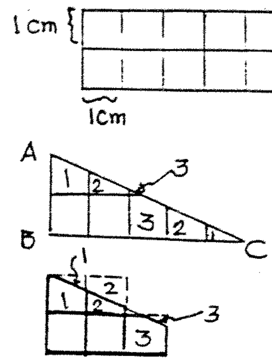


(+) Quadrilateral - is a polygon with four sides and four angles. The sum of all its angles is 360°, or twide that of a triangle.

(++) Parallelogram - a quadrilateral whose pairs of opposite sides are parallel.

3-3.2 Area of Plane Figures

The area of any surface or plane figure is the number of units of area contained in the surface. Thus in the adjoining rectangle, since there are 10 squares(each side of which is 1cm) in it, we say the area of the rectangle is 10 square centimeters. The are of the adjoining right triangle is 5 square centimeters since it contains 2 complete squares and 6 other figures which when taken in pairs form three complete squares, each side of which is 1 cm long.



3-3.2.1 Area of a triangle. If the base of the triangle is denoted by b and the altitude h , then the area of a triangle is found by the formula

$$A = \frac{b \times h}{2}$$

Example: Find the area of a triangle whose base is 10 cm and altitude 4 cm.

Solution: Since b = 10cm and h = 4cm, then the area,

$$A = \frac{10cm \times 4cm}{2} = 20 \text{ square cm}$$

3-3.2.2 Area of a trapezoid. If the bases of a trapezoid is denoted by a and b , and the altitudes as h , then the area of a trapezoid is given by the formula

$$A = \frac{1}{2} \times h \times (a + b)$$

That is, the area of a trapezoid equals one-half the product of the altitude and the sum of its two bases.

Example: Find the area of the trapezoid whose bases are 10 cm and 4 cm and whose altitude is 5 cm.

Solution: Since a = 10 cm, b = 4 cm and h = 5 cm, then by using the formula,

$$A = \tfrac{1}{2} \times 5cm \times (10cm + 4cm) = \tfrac{1}{2} \times 5 \times 14 = 35 \text{ square cm}$$

3-3.2.3 Area of a Rectangle. If the length of the rectangle is denoted by l and the width w, then the area of a rectangle is equal to the number of units of length multiplied by the number of the same units of width, or

$$A = l \times w$$

Example 1: Find the area of a rectangle with a length of 12 cm and a width of 7 cm.

Solution: Since l = 12 cm and w = 7 cm, then

$$A = 12 \text{ cm} \times 7 \text{ cm} = 84 \text{ square cm}$$

Example 2: Find the length of a rectangle if the area is 95 square cm and the width is 5 cm.

Solution: By our formula for finding the area of a rectangle, a = l x w, since the width is given to be 5 cm and the area as 95 sq. cm, then the length is that number of centimeters which when multiplied by 5 cm would give 95 sq.cm. This number is obtained by dividing the value of the area, 95, by the value of the width, 5. Thus,

$$l = A \div w = 95 \div 5 = 19 \text{ cm.}$$

3-3.2.4 Area of a Square. As you already know, a square is a special type of a rectangle with all its sides equal, the l x w in the formula becomes s x s, where s denotes one of the equal sides. The product s x s is written s^2, which is read " s square.". Using this notation, the area of a square is $A = s^2$, where s is the length of a side.

Example: Find the area of a square whose sides are 2.5 cm.

Solution: Since s = 2.5 cm , then

$$A = (2.5)^2 = 6.25 \text{ sq. cm}$$

Example 2: Find the side of a square whose area is 36 sq. sm.

Solution: Since area is known, the side can be found by taking the square root of the area. Thus,

$$s = \sqrt{36 \text{ sq. cm}} = 6 \text{ cm}$$

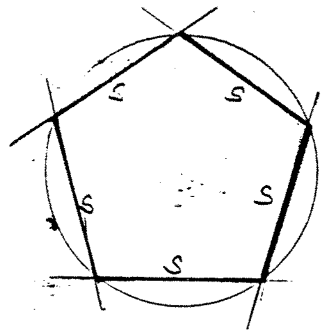
3-4 POLYGON

3-4.1 Characteristics of Polygon

A polygon is a closed figure in a plane. All polygons will have at least three sides. The type of polygons that occurs most often in practical work are regular polygons. In regular polygon all sides are equal in length and all the angles formed by any two adjacent sides have the same measure.

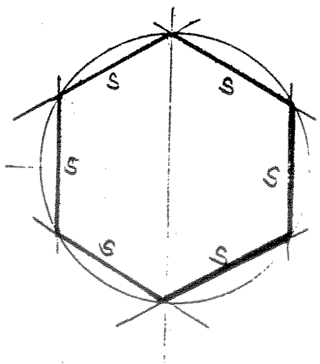
3-4.1.1 Pentagon

A regular five-sided polygon whose sides are all equal in length.



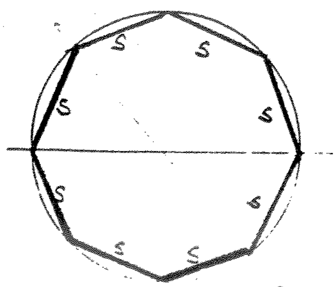
3-4.1.2 Hexagon

A regular six-sided polygon whose six sides are all equal in length.



3-4.1.3 Octagon

A regular eight-sided polygon whose sides are all equal in length.



3-4.2 Area of Polygon

3-4.2.1 Area of any Regular Polygon

Let A - area of any regular polygon
s - length of each side
n - number of equal sides

$$A - \frac{1}{4}\, n\, s^2 \cot \frac{180 \text{ deg}}{n}$$

3-4.2.2 Area of Pentagon

Since a pentagon has 5 equal sides, then its area would be;

$$A - \frac{1}{4}\, n\, s^2 \cot \frac{180 \text{ deg}}{n}$$

$$A - \frac{1}{4}\,(5)\, s^2 \cot \frac{180 \text{ deg}}{5}$$

$$A - 1.25\, s^2\, (1.3764)$$

$$A - 1.72\, s^2$$

3-4.2.3 Area of Hexagon

Since a hexagon has 6 equal sides, then its area would be;

$$A - \frac{1}{4}\, n\, s^2 \cot \frac{180 \text{ deg}}{n}$$

$$A - \frac{1}{4}\,(6)\, s^2 \cot \frac{180 \text{ deg}}{6}$$

$$A - 1.5\, s^2\, (1.732)$$

$$A - 2.598\, s^2$$

3-4.2.4 Area of Octagon

Since an octagon has 8 equal sides, then its area would be;

$$A - \frac{1}{4} n s^2 \cot \frac{180 \text{ deg}}{n}$$

$$A - \frac{1}{4} (8) s^2 \cot \frac{180 \text{ deg}}{8}$$

$$A - 2 s^2 (2.4142)$$

$$A - 4.828 s^2$$

Illustration 1.

Find the area of a regular pentagon whose side is 5.50 in.

Given: s - 5.50 inches
Required: A - ?
Solution:

For a regular pentagon area is equal to $1.72 s^2$.

$$\begin{aligned} A &- 1.72 s^2 \\ &- 1.72 (5.50)^2 \\ &- 1.72 (30.25) \\ &- 52.03 \text{ sq. in.} \end{aligned}$$

Illustration 2.

Find the area of a regular hexagon whose side is 4.0 cm in length. 4.0 cm in length.

Given: s - 4.0 cm
Required: A - ?
Solution:

For a regular hexagon, area is

$A = 2.598 s^2 = 2.598 (4.0)^2 = 2.598 \times 16 = 41.568$

Illustration 3.

find the area ofa regular octagon whose side is 3 ft.

Given: s - 3 ft

Required: A - ?

Solution:

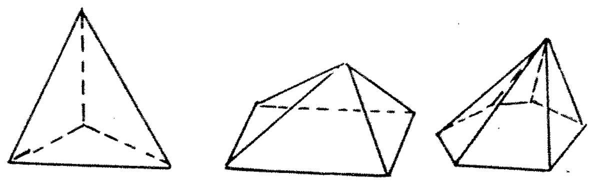
For a regular octagon, area i equal to

$$A = 4.828\ s^2$$
$$= 4.828\ (3.0)^2$$
$$= 4.828\ (9.0)$$
$$= 43.452 \text{ sq. ft.}$$

3.5. Volume of Pyramids and Cones

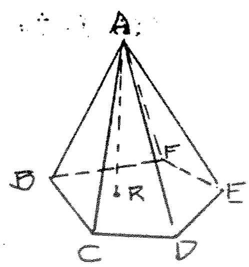
3.5.1. Volume of Pyramide

A pyramid is a solid all of whose faces but one are triangles having a common vertex. This common vertex is called the vertex of the pyramid and the face which does not contain the vertex is the base of the pyramid. The base may be a triangle, a square, or any other polygon.



If a pyramid has a triangular base, it is called a triangular pyramid; if it has a rectangular base, a rectangular pyramid; and so on. Thus a pyramid is named according to what kind of polygon is at the base.

In the adjoining figure, point A is the vertex of the pyramid, and the polygon BCDEF is its base.
The triangular faces each of which has a vertex at A are the lateral faces. The line segment AR drawn from the vertex and perpendicular to the base is called the altitude of the pyramid. The edges of the triangular faces - those edges which are not edges of the base - are called the lateral edges of the pyramid. Thus, AB, AC, AD, AE and AF are the lateral edges of the pyramid under consideration.



If the base of a pyramid is a regular polygon (one in which all sides are equal and all angles are equal), and if the altitude passes through the center of the base, the figure is called a regular pyramid. In a regular pyramid all the lateral edges are equal, and therefore all the lateral faces are isosceles triangles which have the same areas. The altitude drawn from the common vertex to the bases of the isosceles triangles are all equal, and any one of them is called the slant height of the pyramid.

As established in solid geometry, the volume of a pyramid is equal to one third the product of the area of its base by its altitude. In short, if V denotes the volume of a pyramid, h the altitude and B the area of the base, then

$$V = \frac{1}{3} \times B \times h$$

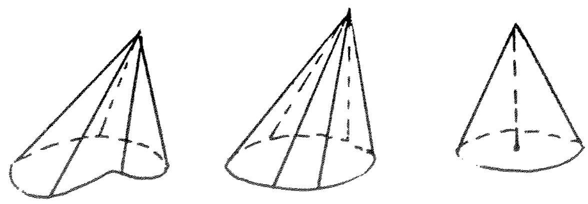
Example : Find the volume of a pyramid whose base has an area of 24 sq.cm and whose altitude is 8 cm.

$$V = \frac{1}{3} \times 24 \text{ sq.cm} \times 8\text{cm} = 64 \text{ cubic cm.}$$

3.5.2. Volume of Cones

Just as a cylinder is similar to a prism, so is the cone to the pyramid. If the triangles on the lateral faces are replaced by a curved surface, then we have a cone.

Like the pyramid, the base is a plane, but while the base of a pyramid is enclosed by a polygon, that of the cone is enclosed by a plane closed curve. Every line segment joining the vertex to a point of the curve at the base lies wholly on the curved surface which is the lateral surface of the cone, and the line segments on the lateral face joining the vertex to any point of the curve at the base are called the elements of the cone.



The perpendicular from the vertex to the plane at the base is the altitude of the cone. If the base is a circle, the cone is called a circular cone.

What we called a regular pyramid corresponds to what we call the right circular cone. This is a cone whose base is a circle, whose elements are equal, and whose altitude passes through the center of the base. Just as the lateral edges of the regular pyramid are all equal, so are all the elements of the right circular cone. However, we did not call any of the lateral edges of the regular pyramid a slant height, but in a right circular cone, any of the elements is called the slant height of the cone.

Like in the pyramid, the volume of the cone is equal to one third the area of the base multiplied by the altitude. In a right circular cone, the base is a circle whose area is $\pi \times r^2$ where r is the radius of the circle. Hence, if we denote the altitude of the cone by h and the volume by V, then

$$V = \frac{1}{3}\pi r^2 h$$

Example : Find the volume of a right circular cone whose altitude is 10 cm and whose base has a radius of 4 cm.

$$V = \frac{1}{3} \times 3.1416 \times 4\text{cm}^2 \times 10 \text{ cm} = 167.55 \text{ cubic cm}$$

Example 2 : A right circular cylinder has an altitude of 8cm and a base whose diameter is equal to 10cm, find the altitude of a right circular cone of equal volume if its base has the same diameter as the cylinder.

Solution : V_c = volume of the cylinder

V_{co} = volume of the cone $\qquad V_c = V_{co}$

d_c = diameter of the cylinder

d_{co} = diameter of the cone $\qquad d_c = d_{co} = 10\text{cm}$

Volume of a right circular cylinder = $V_c = \pi \times r^2 \times h$

$r = \frac{1}{2} d = \frac{10}{2} = 5\text{cm} \qquad h = 8\text{cm}$

$V_c = 3.1416 \times 5^2 \times 8 = 628.32$ cubic cm

Since volume of the cylinder equals the volume of the cone

$V_{co} = 628.32$ cubic cm

Also given that $d_{co} = d_c$ and $r_{co} = 5\text{cm}$

to solve for the altitude of the cone we use the formula

$$h = \frac{(3 \times V)}{\pi r^2}$$

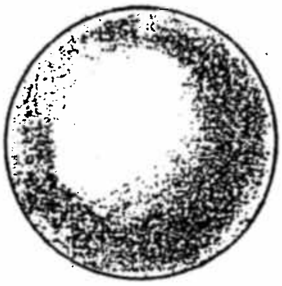
$$h = \frac{(3 \times 628.32 \text{ cm}^3)}{3.1416 \times 5^2} = 24\text{cm}$$ altitude of the right circular cone.

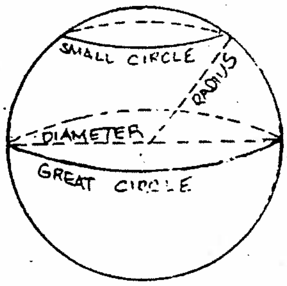
M 3-6 SPHERE 'and HEMISPHERE

3-6.1 Features of Sphere and Hemisphere

3-6.1.1 Sphere

A sphere is a solid bounded by a surface all points of which are equally distant from a point within called the center.





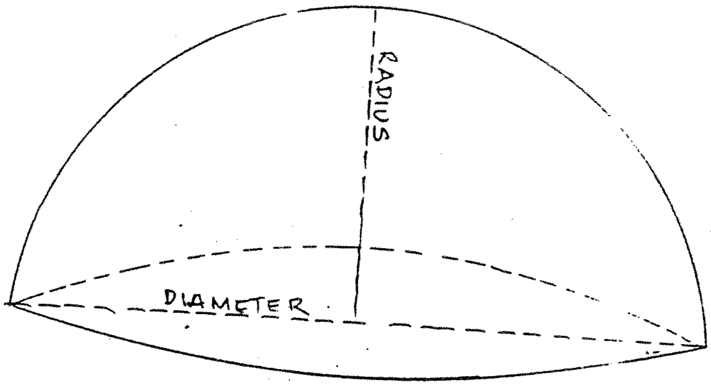
Properties of Sphere

1. Every plane section of a sphere is a circle. If the plane contains a diameter of the sphere, the section is a great circle; otherwise, the section is a small circle.
2. The axis of a circle of a sphere is the diameter of the sphere perpendicular to the plane of the circle.
3. The poles of a circle of a sphere are the ends of its axis.
4. Of two circles cut from a sphere by planes unequally distant from the center, the nearer is the greater.
5. The radius of a great circle is unequal to the radius of the sphere.
6. Two great circle of a sphere bisect each other.
7. All great circles of a sphere are equal.
8. Every great circles of a sphere are equal.
9. The intersection of two spherical surfaces is a circle whose plane is perpendicular to the line joining the centers of the surfaces and whose center is on that line.
10. A plane perpendicular to a radius at its extremity is tangent to the sphere.

11. The shortest line that can be drawn on the surface of a sphere between two points is the shorter arc of the great circle passing through them.

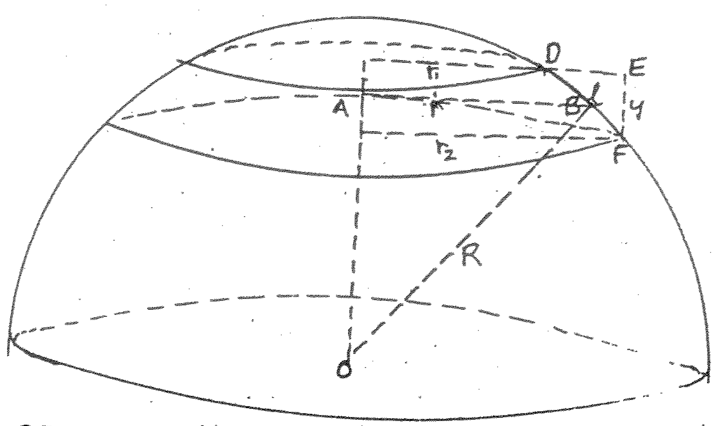
3-6.1.2 Hemisphere

A hemisphere is a solid which is half of a sphere.



3-6 .2 Surface Area of a Sphere and Hemisphere

Consider the hemisphere cut from the sphere of cen center O and radius R. Pass two planes distant y apart and parallel to the base of the hemisphere, cutting the hemisphere in two small circles of radii r_1 and r_2 as shown in the figure.



If we assume arc DF = chord DF = l, the surface of the hemisphere included between these planes is equal to the lateral surface of the inscribed frustum of a right circular cone. This frustum has a slant height l, an altitude y and base radii r_1 and r_2.

Its lateral surface is,

$$S = \left(\frac{2\pi r_1 + 2\pi r_2}{2}\right) l \qquad (1)$$

$$S = 2\pi \left(\frac{r_1 + r_2}{2}\right) l$$

Let B be the midpoint of chord DF. Then OB is perpendicular to chord DF and within the limits of the approximation is equal to the radius R of the sphere. Denote the radius AB of the midsection of the frustum by r. Since r is the mid-section of a trapezoid,

$$r = \frac{r_1 + r_2}{2} \qquad (2)$$

We observe that, since angles AOB and FDE have their sides respectively perpendicular, they have equal and right triangles AOB and FDE are similar. Therefore

$$\frac{r}{R} = \frac{y}{l}$$

$$r = \frac{Ry}{l} \qquad (3)$$

Substituting in equation (3) the value of r from equation 9 (2) we get

$$\frac{r_1 + r_2}{2} = \frac{Ry}{l}$$

Substituting this value of $\frac{r_1 + r_2}{2}$ in formula (1) we obtain

$$S = 2\pi Ry$$

By thinking of a sphere as being formed by an indefinitely large number of these frustums, the sum of whose altitudes is 2R, it is evident that the formula for the surface of a sphere of radius R is

$$S = 2\pi R(2R),$$

Generally, the area of the surface of a sphere is equal to the area of four of its great circles.

$$\boxed{S = 4\pi R^2}$$

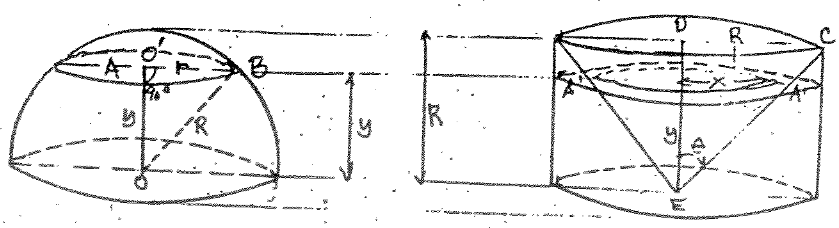
Since Hemisphere is half of a sphere therefore, the area of the surface of a hemisphere is one half the surface area of a sphere.

$$S = \frac{1}{2}\left(4\pi R^2\right)$$

$$\boxed{S = 2\pi R^2}$$

3-6.3 Volume of Sphere and Hemisphere

Consider the hemisphere cut from the sphere of c center O and radius R. Compare this hemisphere with the solid which results from removing a right circular cone of base radius R and altitude R from a right circular cylinder of the same base and altitude, as shown in the figure.



Place the two solids so that their bases lie in the same plane parallel to and distant y from the bases, cutting the hemisphere in small circle A and the other solid in section A'(area bounded by two concentric circles as shown). Denote the radius of circle A by r, the inner radius of section A' by x (the outer radius of section A' is obviously R), and write

$$A = \pi r^2 \qquad (1)$$

$$A' = \pi (R^2 - x^2) \qquad (2)$$

Since the legs of right triangle CDE are each R, $\theta = 45^o$ when $x = y$.
Applying the Pythagorrean theorem to right triangle OO'B, we have

$$r^2 = R^2 - y^2$$

Substituting this value of r^2 in equation (1) and putting $x = y$ in equation (2), we obtain

$$A = \pi (R^2 - y^2)$$

and

$$A' = \pi (R^2 - y^2)$$

From these equations we have

$$A = A'$$

Since the altitude of each solid is equal to R and since $A = A'$, it follows from Cavalieri's theorem that the volumes of the two solids are equal. But, denoting the volume of the constructed solid by V_1, we have

V_1 = volume of cylinder - volume of cone

$$V_1 = (\pi R^2)R - \frac{1}{3}(\pi R^2)R$$

$$V_1 = \frac{2}{3}\pi R^3$$

Therefore the volume of the hemisphere is

$$\boxed{V = \frac{2}{3} \pi R^3}$$

Since hemisphere is twice the sphere, hence the volume of a sphere of radius R is

$$V = 2\left(\frac{2}{3} \pi R^3\right)$$

$$\boxed{V = \frac{4}{3} \pi R^3}$$

Illustration 1:

Find the surface area of a sphere 20 cm in diameter.

Given: diameter of sphere = 20 cm
Req. red: surface area of the sphere
Solution:

$$\begin{aligned} S &= 4\pi R^2 \\ &= 4\,(3.1416)\,(20)^2 \\ &= 4\,(3.1416)\,(400) \\ &= 5026.56 \text{ cm}^2 \end{aligned}$$

Illustration 2:

Find the volume of a sphere whose RADIUS is equal to 15 inches

Given: radius of sphere = 15 in.
Required: volume of the sphere
Solution:

$$\begin{aligned} V &= \frac{4}{3} \pi R^3 \qquad \text{since } R = \frac{1}{2} D \\ &= \frac{4}{3} \pi \left(\frac{1}{2} D \right)^3 \\ &= \frac{4}{3} (3.1416) \left(\frac{1}{8} \right)(15)^3 \\ &= 1767.15 \text{ in}^3 \end{aligned}$$

Illustration 3:

A hemisphere is 3'4" in diameter. Find its surface area, in square inches.

Given: diameter of hemisphere = 3'4"

Required: Surface area of the hemisphere

Solution:

$$\text{diameter} = (3 \times 12) + 4$$
$$= 40 \text{ in.}$$
$$\text{radius} = \tfrac{1}{2} D$$
$$= \tfrac{1}{2} (40)$$
$$= 20 \text{ in.}$$

$$\text{Surface area } (S) = 2\pi R^2$$
$$= 2\,(3.1416)\,(20)^2$$
$$= 2\,(3.1416)\,(400)$$
$$= 2513.28 \text{ in}^2$$

Illustration 4:

Find the volume of a hemisphere whose radius is 1.375 meter.

Given: Radius of hemisphere = 1.375 meter

Required: volume of the hemisphere

Solution:

$$V = \tfrac{2}{3}\pi R^3$$
$$= \tfrac{2}{3}\,(3.1416)\,(1.375)^3$$
$$= \tfrac{2}{3}\,(3.1416)\,(2.5996)$$
$$= 5.4446 \text{ meter}^3$$